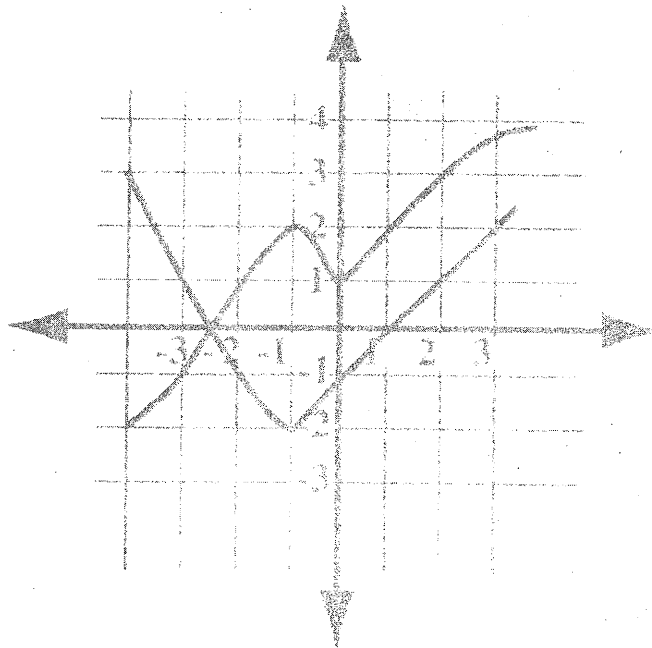


Statement on Competencies in Mathematics

Expected of Entering College Students

Intersegmental Committee of the Academic Senates
of the California Community Colleges,
the California State University and the University of California



Standards in Mathematics for California High School Graduates

Sponsored by the
California Education Round Table

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BARRY MUNITZ
CHANCELLOR

August 1997

Dear Colleague:

Two documents have been approved recently describing standards in mathematics for secondary students in California. The first of these is the *Statement on Competencies in Mathematics Expected of Entering College Students*, written by postsecondary and secondary faculty under the aegis of the Intersegmental Committee of the Academic Senates, which represents the faculties of the California Community Colleges, the California State University, and the University of California. Closely related are the *Standards in Mathematics for California High School Graduates* developed in a collaborative process including faculty, teachers, and public representatives and approved by the California Education Round Table earlier this year.

The Round Table document proposes mathematical content standards that all California's students should meet in order to graduate from high school. For high school graduates who go on to college, higher standards are described in the *Statement on Competencies*. As you'll see, these two documents are aligned in several ways. The standards for high school graduation are included within the expectations for entering college students. The documents are also aligned in their perspectives, as they both represent a moderate stance in the current debate between mathematics reform and traditional approaches. Finally, both documents were written collaboratively by representative committees that considered a breadth of opinions from around the state.

These two documents are bundled here because together they provide strong direction for secondary mathematics education in California. They make it clear that the mathematical expectations for students who plan to go to college are an extension of the standards for students to graduate from high school. Taken together they should be useful to teachers, school administrators, school board members, parents, students, and members of the community—to anyone interested in strengthening secondary mathematics education in California.

Sincerely,

Barry Munitz, Chair
California Education Round Table *and*
Chancellor, The California State University

James M. Highsmith, Chair
Intersegmental Committee of the
Academic Senates *and*
Chair, Academic Senate CSU

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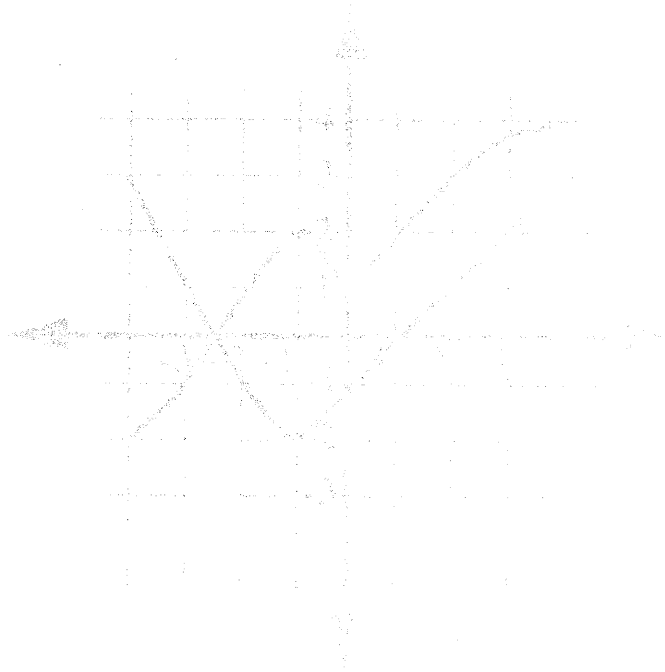
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Statement on Competencies in Mathematics Expected of Entering College Students



Intersegmental Committee of the Academic Senates
of the
California Community Colleges,
the California State University and the University of California
1996/97



June 1997

Statement on Competencies in Mathematics Expected of Entering College Students

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INTERSEGMENTAL COMMITTEE OF THE ACADEMIC SENATES
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1996/97

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June 30, 1997

Dear Colleague:

Here is the "1997 Statement on Competencies in Mathematics Expected of Entering College Students." It provides a clear description of the mathematics that students entering public colleges and universities in California need to know and be able to do in order to be successful in their postsecondary studies.

This Statement is the result of a remarkable collaboration among secondary mathematics teachers and college and university faculty. It has benefited from the many comments and suggestions from people throughout California who responded to the review draft distributed in 1996. In contrast to the sometimes shrill debates regarding mathematics education, it represents a moderate and broad consensus from higher education.

The Statement was sponsored and adopted by the Academic Senates of the California Community Colleges, the California State University, and the University of California, and by their voluntary organization, ICAS, the Intersegmental Committee of the Academic Senates. It is the official recommendation of California public higher education faculty to secondary (and primary) teachers about the mathematics preparation their students are expected to achieve.

The knowledge of mathematics described in the Statement as necessary for all college-bound students is a minimum—many will need more extensive preparation. Students who will major in areas that make significant use of mathematics (such as science, engineering, mathematics, business, and social science) need a stronger mathematical background including preparation for calculus.

The brevity of this Statement on mathematics competencies is intentional. Please read it and use it in its entirety. Students need the competencies described in each section.

Please discuss the Statement with your colleagues and assist us in its dissemination. ICAS grants permission for reproduction of the "1997 Statement on Competencies in Mathematics Expected of Entering College Students" for educational purposes.

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CSU Chair
Academic Senate

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UC Chair
Academic Council

Statement on Competencies in Mathematics Expected of Entering College Students

Introduction

The goal of this *Statement on Competencies in Mathematics Expected of Entering College Students* is to provide a clear and coherent message about the mathematics that students need to know and to be able to do to be successful in college. While this is written especially for the secondary mathematics teachers, it should be useful for anyone who is concerned about the preparation of California's students for college. This represents an effort to be realistic about the skills, approaches, experiences, and subject matter that make up an appropriate mathematical background for entering college students.

The first section describes some characteristics that identify the student who is properly prepared for college courses that are quantitative in their approach. The second section describes the background in technology, such as calculators, that college students should have. The third section describes the subject matter that is an essential part of the background for all entering college students, as well as describing what is the essential background for students intending quantitative majors. Among the descriptions of subject matter there are sample problems. These are intended to clarify the descriptions of subject matter and to be representative of the appropriate level of understanding. The sample problems do not cover all of the mathematical topics identified.

No section of this Statement should be ignored. Students need the approaches, attitudes, and perspectives on mathematics described in the first section. Students need the experiences with technology described in the second section. And students need extensive skills and knowledge in the subject matter areas described in the third section. Inadequate attention to any of these components is apt to disadvantage the student in ways that impose a serious impediment to success in college. Nothing less than a balance among these components is acceptable for California's students.

The discussion in this document of the mathematical competencies expected of entering college students is predicated on this highlighted basic recommendation.

For proper preparation for baccalaureate level course work, all students should be enrolled in a mathematics course in every semester of high school. It is particularly important that students take mathematics courses in their senior year of high school, even if they have completed three years of college preparatory mathematics by the end of their junior year. Experience has shown that students who take a hiatus from the study of mathematics in high school are very often unprepared for courses of a quantitative nature in college and are unable to continue in these courses without remediation in mathematics.

Section 1

Approaches to Mathematics

This section enumerates characteristics of entering college students who have the mathematical maturity to be successful in a first college mathematics course, and in other college courses that are quantitative in their approach. These characteristics are described primarily in terms of how students approach mathematical problems. The second part of this section provides suggestions to secondary teachers of ways to present mathematics that will help their students to develop these characteristics.

Part 1

Dispositions of Well-prepared Students Toward Mathematics

Crucial to their success in college is the way in which students encounter the challenges of new problems and new ideas. From their high school mathematics courses students should have gained certain approaches, attitudes, and perspectives:

- A view that mathematics makes sense—students should perceive mathematics as a way of understanding, not as a sequence of algorithms to be memorized and applied.
- An ease in using their mathematical knowledge to solve unfamiliar problems in both concrete and abstract situations—students should be able to find patterns, make conjectures, and test those conjectures; they should recognize that abstraction and generalization are important sources of the power of mathematics; they should understand that mathematical structures are useful as representations of phenomena in the physical world; they should consistently verify that their solutions to problems are reasonable.
- A willingness to work on mathematical problems requiring time and thought, problems that aren't solved by merely mimicking examples that have already been seen—students should have enough genuine success in solving such problems to be confident, and thus to be tenacious, in their approach to new ones.
- A readiness to discuss the mathematical ideas involved in a problem with other students and to write clearly and coherently about mathematical topics—students should be able to communicate their understanding of mathematics with peers and teachers using both formal and natural languages correctly and effectively.
- An acceptance of responsibility for their own learning—students should realize that their minds are their most important mathematical resource, and that teachers and other students can help them to learn but can't learn for them.
- The understanding that assertions require justification based on persuasive arguments, and an ability to supply appropriate justifications—students should habitually ask "Why?" and should have a familiarity with reasoning at a variety of levels of formality, ranging from concrete examples through informal arguments using words and pictures to precise structured presentations of convincing arguments.
- An openness to the use of appropriate technology, such as graphing calculators and computers, in solving mathematical problems and the attendant awareness of the limitations of this technology—students should be able to make effective use of the technology, which includes the ability to determine when technology will be useful and when it will not be useful.
- A perception of mathematics as a unified field of study—students should see interconnections among various areas of mathematics, which are often perceived as distinct.

Part 2

Aspects of Mathematics Instruction to Foster Student Understanding and Success

There is no best approach to teaching, not even an approach that is effective for all students, or for all instructors. One criterion that should be used in evaluating approaches to teaching mathematics is the extent to which they lead to the development in the student of the dispositions, concepts, and skills that are crucial to success in college. It should be remembered that in the mathematics classroom, time spent focused on mathematics is crucial. The activities and behaviors that can accompany the learning of mathematics must not become goals in themselves—understanding of mathematics is always the goal.

While much has been written recently about approaches to teaching mathematics, as it relates to the preparation of students for success in college, there are a few aspects of mathematics instruction that merit emphasis here.

Modeling Mathematical Thinking

Students are more likely to become intellectually venturesome if it is not only expected of them, but if their classroom is one in which they see others, especially their teacher, thinking in their presence. It is valuable for students to learn with a teacher and others who get excited about mathematics, who work as a team, who experiment and form conjectures. They should learn by example that it is appropriate behavior for people engaged in mathematical exploration to follow uncertain leads, not always to be sure of the path to a solution, and to take risks. Students should understand that learning mathematics is fundamentally about inquiry and personal involvement.

Solving Problems

Problem solving is the essence of mathematics. Problem solving is not a collection of specific techniques to be learned; it cannot be reduced to a set of procedures. Problem solving is taught by giving students appropriate experience in solving unfamiliar problems, by then engaging them in a discussion of their various attempts at solutions, and by reflecting on these processes. Students entering college should have had successful experiences solving a wide variety of mathematical problems. The goal is the development of open, inquiring, and demanding minds. Experience in solving problems gives students the confidence and skills to approach new situations creatively, by modifying, adapting, and combining their mathematical tools; it gives students the determination to refuse to accept an answer until they can explain it.

Developing Analytic Ability and Logic

A student who can analyze and reason well is a more independent and resilient student. The instructional emphasis at all levels should be on a thorough understanding of the subject matter and the development of logical reasoning. Students should be asked “Why?” frequently enough that they anticipate the question, and ask it of themselves. They should be expected to construct compelling arguments to explain why, and to understand a proof comprising a significant sequence of implications. They should be expected to question and to explore why one statement follows from another. Their understandings should be challenged with questions that cause them to further examine their reasoning. Their experience with mathematical proof should not be limited to the format of a two-column proof; rather, they should see, understand, and construct proofs in various formats throughout their course work. A classroom full of discourse and interaction that focuses on reasoning is a classroom in which analytic ability and logic are being developed.

Experiencing Mathematics in Depth

Students who have seen a lot but can do little are likely to find difficulty in college. While there is much that is valuable to know in the breadth of mathematics, a shallow but broad mathematical experience does not develop the sort of mathematical sophistication that is most valuable to students in college. Emphasis on coverage of too many topics can trivialize the mathematics that awaits the students, turn the study of mathematics into the memorization of discrete facts and skills, and divest students of their curiosity. By delving deeply into well-chosen areas of mathematics, students develop not just the self-confidence but the ability to understand other mathematics more readily, and independently.

Appreciating the Beauty and Fascination of Mathematics

Students who spend years studying mathematics yet never develop an appreciation of its beauty are cheated of an opportunity to become fascinated by ideas that have engaged individuals and cultures for thousands of years. While applications of mathematics are valuable for motivating students, and as paradigms for their mathematics, an appreciation for the inherent beauty of mathematics should also be nurtured, as mathematics is valuable for more than its utility. Opportunities to enjoy mathematics can make the student more eager to search for patterns, for connections, for answers. This can lead to a deeper mathematical understanding, which also enables the student to use mathematics in a greater variety of applications. An appreciation for the aesthetics of mathematics should permeate the curriculum and should motivate the selection of some topics.

Building Confidence

For each student, successful mathematical experiences are self-perpetuating. It is critical that student confidence be built upon genuine successes—false praise usually has the opposite effect. Genuine success can be built in mathematical inquiry and exploration. Students should find support and reward for being inquisitive, for experimenting, for taking risks, and for being persistent in finding solutions they fully understand. An environment in which this happens is more likely to be an environment in which students generate confidence in their mathematical ability.

Communicating

While solutions to problems are important, so are the processes that lead to the solutions and the reasoning behind the solutions. Students should be able to communicate all of this, but this ability is not quickly developed. Students need extensive experiences in oral and written communication regarding mathematics, and they need constructive, detailed feedback in order to develop these skills. Mathematics is, among other things, a language, and students should be comfortable using the language of mathematics. The goal is not for students to memorize an extensive mathematical vocabulary, but rather for students to develop an ease in carefully and precisely discussing the mathematics they are learning. Memorizing terms that students don't use does not contribute to their mathematical understanding. However, using appropriate terminology so as to be precise in communicating mathematical meaning is part and parcel of mathematical reasoning.

Becoming Fluent in Mathematics

To be mathematically capable, students must have a facility with the basic techniques of mathematics. There are necessary skills and knowledge that students must routinely exercise without hesitation (see Appendix A). Mathematics is the language of the sciences, and thus fluency in this language is a basic skill. College mathematics classes require that students bring with them an ease with the standard skills of mathematics that allows them to focus on the ideas and not become lost in the details. However, this level of internalization of mathematical skills should not be mistaken for the only objective of secondary mathematics education. Student understanding of mathematics is the goal. In developing a skill, students first must develop an understanding. Then as they use the skill in different contexts, they gradually wean themselves from thinking about it deeply each time, until its application becomes routine. But their understanding of the mathematics is the map they use whenever they become disoriented in this process. The process of applying skills in varying and increasingly complex applications is one of the ways that students not only sharpen their skills, but also reinforce and strengthen their understanding. Thus, in the best of mathematical environments, there is no dichotomy between gaining skills and gaining understanding. A curriculum that is based on depth and problem solving can be quite effective in this regard provided that it focuses on appropriate areas of mathematics.

Section 2

Technology

The pace at which advances are made in technology, and the surprising ways in which mathematics pedagogy and curriculum change in response to those advances, make it impossible to anticipate what technological experiences and skills students will need for success in college in the coming years. Also, the diversity of responses to technology among the college mathematics courses in California further impede the development of a clear statement on the appropriate technological background for entering college students. But the general directions are discernible. The past has shown us that scientific calculators make many problems accessible to students that previously were not because of excessive computation. More recently, we've seen that students can use the graphing capabilities of calculators to deepen their understanding of functions. And now the advent of hand-held calculators that perform symbolic algebra computations will certainly have a major impact on the instruction in algebra and more advanced courses.

From all of this, it is clear that entering college students must have availed themselves of opportunities presented by technology. The kind of graphing calculator or computer software preferred at different institutions, by different instructors, in different courses, at different times will of course vary. So, student experiences should not focus on the intricacies of a specific device so much as on the use of technology as a valuable tool in many aspects of their mathematics courses. Entering college students should have considerable experience in the following areas:

- Deciding when to use technology. Students should be able to determine what algebraic or geometric manipulations are necessary to make best use of the calculator. At the same time, they should also be able to determine for themselves when using a calculator, for example, might be advantageous in solving a problem.
- Dealing with data. Students should work on problems posed around real data and involving significant calculations. With repeated applications requiring computation, they can gain skill in estimation, approximation, and the ability to tell if a proposed solution is reasonable. Students should find opportunities to work with data in algebra, geometry, and statistics.
- Checking their calculations. Whenever possible students should use a calculator with a multi-line screen so that they are able to review the input to the calculator and to determine whether any errors have been made.
- Representing problems geometrically. Students should be able to use graphing calculators as a tool to represent functions and to develop a deeper understanding of domain, range, arithmetic operations on functions, inverse functions, and function composition.
- Experimenting, making conjectures, and finding counterexamples. Students should be comfortable using technology to check their guesses, to formulate revised guesses, and to make conjectures based on these results. They should also challenge conjectures, and find counterexamples. Where possible, they should use tools such as geometric graphing utilities to make and test geometric conjectures and to provide counterexamples.

Section 3

Subject Matter

Decisions about the subject matter for secondary mathematics courses are often difficult, and are too easily based on tradition and partial information about the expectations of the colleges. What follows is a description of mathematical areas of focus that are (1) essential for all entering college students; (2) desirable for all entering college students; (3) essential for college students to be adequately prepared for quantitative majors; and (4) desirable for college students who intend quantitative majors. This description of content will in many cases necessitate adjustments in a high school mathematics curriculum, generally in the direction of deeper study in the more important areas, at the expense of some breadth of coverage.

Sample problems have been included to indicate the appropriate level of understanding for some areas. The problems included do not cover all of the mathematical topics described, and many involve topics from several areas. Entering college students working independently should be able to solve problems like these in a short time—less than half an hour for each problem included. Students must also be able to solve more complex problems requiring significantly more time.

Part 1

Essential Areas of Focus for All Entering College Students

What follows is a summary of the mathematical subjects that are an essential part of the knowledge base and skill base for all students who enter higher education. Students are best served by deep mathematical experiences in these areas. This is intended as a brief compilation of the truly essential topics, as opposed to topics to which students should have been introduced but need not have mastered. The skills and content knowledge that are prerequisite to high school mathematics courses are of course still necessary for success in college, although they are not explicitly mentioned here. Relative to traditional practice, topics and perspectives are described here as appropriate for increased emphasis (which does not mean paramount importance) and for decreased emphasis (which does not mean elimination).

- *Variables, Equations, and Algebraic Expressions:* Algebraic symbols and expressions; evaluation of expressions and formulas; translation from words to symbols; solutions of linear equations and inequalities; absolute value; powers and roots; solutions of quadratic equations; solving two linear equations in two unknowns including the graphical interpretation of a simultaneous solution. Increased emphasis should be placed on algebra both as a language for describing mathematical relationships and as a means for solving problems, while decreased emphasis should be placed on interpreting algebra as merely a set of rules for manipulating symbols.

The braking distance of a car (how far it travels after the brakes are applied until it comes to a stop) is proportional to the square of its speed.

Write a formula expressing this relationship and explain the meaning of each term in the formula.

If a car traveling 50 miles per hour has a braking distance of 105 feet, then what would its braking distance be if it were traveling 60 miles per hour?

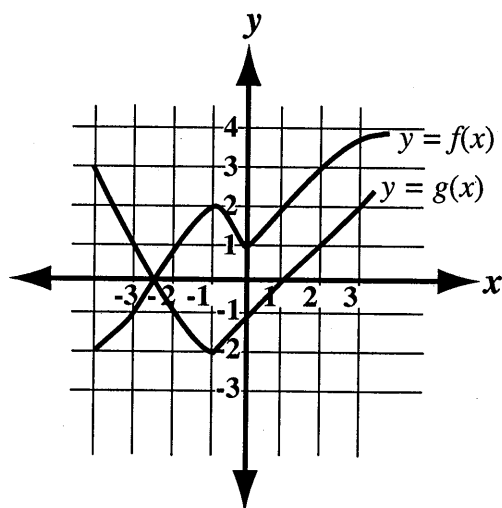
Solve for x and give a reason for each step: $\frac{2}{3x+1} + 2 = \frac{2}{3}$

United States citizens living in Switzerland must pay taxes on their income to both the United States and to Switzerland. Suppose that the United States tax is 28% of their taxable income after deducting the tax paid to Switzerland. Suppose that the tax paid to Switzerland is 42% of their taxable income after deducting the tax paid to the United States. If a United States citizen living in Switzerland has a taxable income of \$75,000, how much tax must that citizen pay to each of the two countries? Find these values in as many different ways as you can; try to find ways both using and not using graphing calculators. Explain the methods you use.

■ *Families of Functions and Their Graphs: Applications; linear functions; quadratic and power functions; exponential functions; roots; operations on functions and the corresponding effects on their graphs; interpretation of graphs; function notation; functions in context, as models for data. Increased emphasis should be placed on various representations of functions—using graphs, tables, variables, words—and on the interplay among the graphical and other representations, while decreased emphasis should be placed on repeated manipulations of algebraic expressions.*

Car dealers use the “rule of thumb” that a car loses about 30% of its value each year. Suppose that you bought a new car in December 1995 for \$20,000. According to this “rule of thumb,” what would the car be worth in December 1996? In December 1997? In December 2005?

Develop a general formula for the value of the car t years after purchase.



- Which is larger, $f(-3)$ or $f(3)$?
- Which among the following three quantities is the largest?
 $f(-1) - g(-1)$?
 $f(0) - g(0)$?
 $f(1) - g(1)$?
- For which values of x does $g(x) = f(-3)$?
- Find a value of x for which $f(x) = f(x+2)$

Find a quadratic function of x that has zeroes at $x = -1$ and $x = 2$.

Find a cubic function of x that has zeroes at $x = -1$ and $x = 2$ and nowhere else.

- Geometric Concepts:* Distances, areas, and volumes, and their relationship with dimension; angle measurement; similarity; congruence; lines, triangles, circles, and their properties; symmetry; Pythagorean Theorem; coordinate geometry in the plane, including distance between points, midpoint, equation of a circle; introduction to coordinate geometry in three dimensions; right angle trigonometry. Increased emphasis should be placed on developing an understanding of geometric concepts sufficient to solve unfamiliar problems and an understanding of the need for compelling geometric arguments, while decreased emphasis should be placed on memorization of terminology and formulas.

A contemporary philosopher wrote that in 50 days the earth traveled approximately 40 million miles along its orbit and that the distance between the positions of the earth at the beginning and the end of the 50 days was approximately 40 million miles. Discuss any errors you can find in these conclusions or explain why they seem to be correct. You may approximate the earth's orbit by a circle with radius 93 million miles.

ABCD is a square and the midpoints of the sides are E, F, G, and H. $AB = 10$ in. Use at least two different methods to find the area of parallelogram AFCH.

Two trees are similar in shape, but one is three times as tall as the other. If the smaller tree weighs two tons, how much would you expect the larger tree to weigh?

Suppose that the bark from these trees is broken up and placed into bags for landscaping uses. If the bark from these trees is the same thickness on the smaller tree as the larger tree, and if the larger tree yields 540 bags of bark, how many bags would you expect to get from the smaller tree?

A sheet of paper can be rolled lengthwise to make a cylinder, or it can be rolled widthwise to make a different cylinder.

Without computing the volumes of the two cylinders, predict which will have the greater volume, and explain why you expect that.

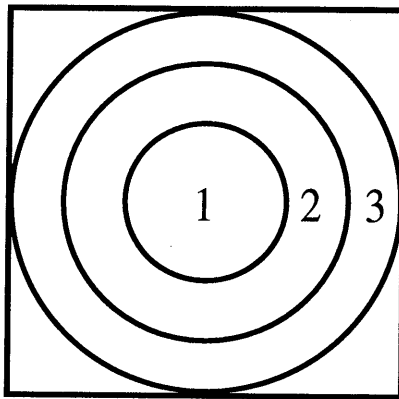
Find the volumes of the two cylinders to see if your prediction was correct.

If the cylinders are to be covered top and bottom with additional paper, which way of rolling the cylinder will give the greater total surface area?

■ *Probability:* Counting (permutations and combinations, multiplication principle); sample spaces; expected value; conditional probability; area representations of probability. Increased emphasis should be placed on a conceptual understanding of discrete probability, while decreased emphasis should be placed on aspects of probability that involve memorization and rote application of formulas.

If you take one jelly bean from a large bin containing 10 lbs. of jelly beans, the chance that it is cherry flavored is 20 percent. How many more pounds of cherry jelly beans would have to be mixed into the bin to make the chance of getting a cherry one 25 percent?

A point is randomly illuminated on a computer game screen that looks like the figure shown below.



The radius of the inner circle is 3 inches; the radius of the middle circle is 6 inches; the radius of the outer circle is 9 inches.

What is the probability that the illuminated point is in region 1?

What is the probability that the illuminated point is in region 1 if you know that it isn't in region 2?

A fundraising group sells 1000 raffle tickets at \$5 each. The first prize is an \$1,800 computer. Second prize is a \$500 camera and the third prize is \$300 cash. What is the expected value of a raffle ticket?

Five friends line up at a movie theater. What is the probability that Mary and Mercedes are standing next to each other?

- Data Analysis and Statistics:** Presentation and analysis of data; mean, median and standard deviation; representative samples; using lines to fit data and make predictions. Increased emphasis should be placed on organizing and describing data and making predictions based on the data, with common sense as a guide, while decreased emphasis should be placed on aspects of statistics that are learned as algorithms without an understanding of the underlying ideas.

The table at the right shows the population of the USA in each of the last five censuses.

Make a scatter plot of this data and draw a line on your scatter plot that fits this data well.

Find an equation for your line, and use this equation to predict what the population might be in the year 2000.

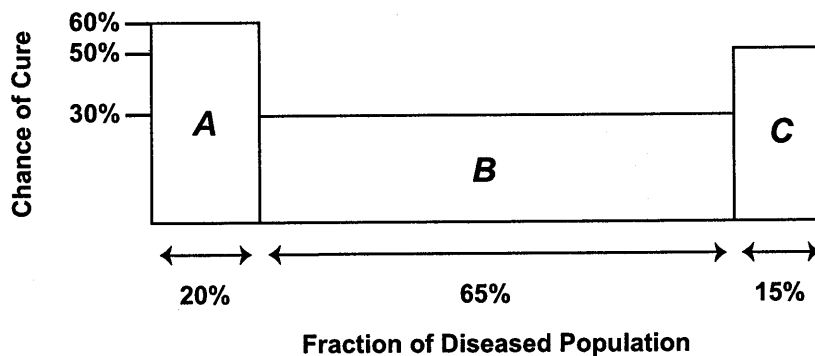
Plot that predicted point on your graph and see if it seems reasonable.

What is the slope of your line?

Write a sentence that describes to someone who might not know about graphs and lines what the meaning of the slope is in terms involving the USA population.

Year	USA Population (millions)
1950	152.3
1960	180.7
1970	205.1
1980	227.7
1990	249.9

The results of a study of the effectiveness of a certain treatment for a blood disease are summarized in the chart shown below. The blood disease has three types, A, B, and C. The cure rate for each of the types is shown vertically on the chart. The percentage of diseased persons with each type of the disease is shown horizontally on the same chart.



For which type of the disease is the treatment most effective?

From which type of the disease would the largest number of patients be cured by the treatment?

What is the average cure rate of this treatment for all of the persons with the disease?

Find the mean and standard deviation of the following seven numbers:

4 12 5 6 8 5 9

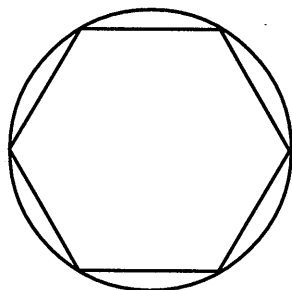
Make up another list of seven numbers with the same mean and a smaller standard deviation.

Make up another list of seven numbers with the same mean and a larger standard deviation.

- *Argumentation and Proof:* Mathematical implication; hypotheses and conclusions; direct and indirect reasoning; inductive and deductive reasoning. Increased emphasis should be placed on constructing and recognizing valid mathematical arguments, while decreased emphasis should be placed on mathematical proofs as formal exercises.

Select any odd number, then square it, and then subtract one. Must the result always be even? Write a convincing argument.

Use the perimeter of a regular hexagon inscribed in a circle to explain why $\pi > 3$.



Does the origin lie inside of, outside of, or on the geometric figure whose equation is $x^2 + y^2 - 10x + 10y - 1 = 0$? Explain your reasoning.

Part 2

Desirable Areas of Focus for All Entering College Students

What follows is a brief summary of some of the mathematical subjects that are a **desirable** part of the mathematical experiences for all students who enter higher education. No curriculum would include study in all of these areas, as that would certainly be at the expense of opportunities for deep explorations in selected areas. But these areas provide excellent contexts for the approaches to teaching suggested in Section I, and any successful high school mathematics program will include some of these topics. The emphasis here is on enrichment and on opportunities for student inquiry.

- *Discrete Mathematics:* Graph theory; coding theory; voting systems; game theory; decision theory.
- *Sequences and Series:* Geometric and arithmetic sequences and series; the Fibonacci sequence; recursion relations.

- *Geometry*: Transformational geometry, including rotations, reflections, translations, and dilations; tessellations; solid geometry; three-dimensional coordinate geometry, including lines and planes.
- *Number Theory*: Prime numbers; prime factorization; rational and irrational numbers; triangular numbers; Pascal's triangle; Pythagorean triples.

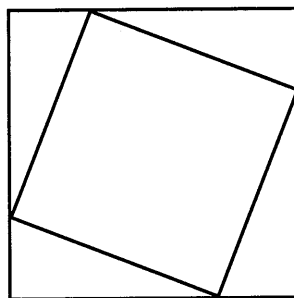
Part 3

Essential Areas of Focus for Students in Quantitative Majors

What follows is a brief summary of the mathematical subjects that are an essential part of the knowledge base and skill base for students to be adequately prepared for quantitative majors. Students are best served by deep mathematical experiences in these areas. The skills and content knowledge listed above as essential for all students entering college are of course also essential for these students—moreover, students in quantitative majors must have a deeper understanding of and a greater facility with those areas.

- *Variables, Equations, and Algebraic Expressions*: Solutions to systems of equations, and their geometrical interpretation; solutions to quadratic equations, both algebraic and graphical; the correspondence between roots and factors of polynomials; the binomial theorem.

In the figure shown to the right, the area between the two squares is 11 square inches. The sum of the perimeters of the two squares is 44 inches. Find the length of a side of the larger square.



Determine the middle term in the binomial expansion of $(x - \frac{2}{x})^{10}$.

- *Functions:* Logarithmic functions, their graphs, and applications; trigonometric functions of real variables, their graphs, properties including periodicity, and applications; basic trigonometric identities; operations on functions, including addition, subtraction, multiplication, reciprocals, division, composition, and iteration; inverse functions and their graphs; domain and range.

Which of the following functions are their own inverses? Use at least two different methods to answer this, and explain your methods.

$$f(x) = \frac{2}{x}$$

$$g(x) = x^3 + 4$$

$$h(x) = \frac{2 + \ln(x)}{2 - \ln(x)}$$

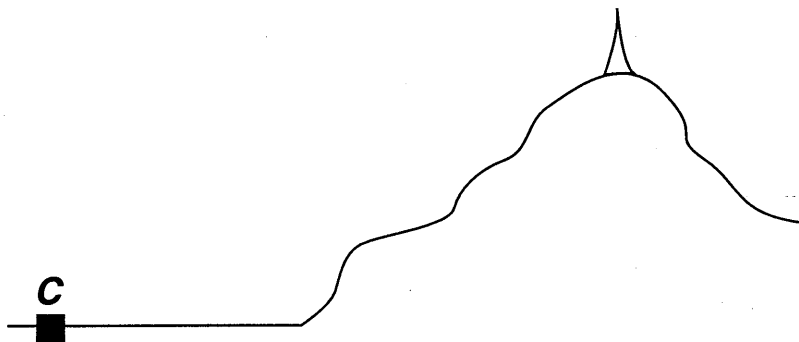
$$j(x) = \sqrt[3]{\frac{x^3 + 1}{x^3 - 1}}$$

Scientists have observed that living matter contains, in addition to carbon, C12, a fixed percentage of a radioactive isotope of carbon, C14. When the living material dies, the amount of C12 present remains constant, but the amount of C14 decreases exponentially with a half life of 5,550 years. In 1965, the charcoal from cooking pits found at a site in Newfoundland used by Vikings was analyzed and the percentage of C14 remaining was found to be 88.6 percent. What was the approximate date of this Viking settlement?

Find all quadratic functions of x that have zeroes at $x = -1$ and $x = 2$.

Find all cubic functions of x that have zeroes at $x = -1$ and $x = 2$ and nowhere else.

A cellular phone system relay tower is located atop a hill. You have a transit and a calculator. You are standing at point C. Assume that you have a clear view of the base of the tower from point C, that C is at sea level, and that the top of the hill is 2000 ft. above sea level.



Describe a method that you could use for determining the height of the relay tower, without going to the top of the hill.

Next choose some values for the unknown measurements that you need in order to find a numerical value for the height of the tower, and find the height of the tower.

- *Geometric Concepts:* Two- and three-dimensional coordinate geometry; locus problems; polar coordinates; vectors; parametric representations of curves.

Find any points of intersection (first in polar coordinates and then in rectangular coordinates) of the graphs of $r=1+\sin \theta$ and the circle of radius $\frac{3}{2}$ centered about the origin. Verify your solutions by graphing the curves.

Find any points of intersection (first in polar coordinates and then in rectangular coordinates) of the graphs of $r=1+\sin \theta$ and the line with slope 1 that passes through the origin. Verify your solutions by graphing the curves.

Marcus is in his backyard, and has left his stereo and a telephone 24 feet apart. He can't move the stereo or the phone, but he knows from experience that in order to hear the telephone ring, he must be located so that the stereo is at least twice as far from him as the phone. Draw a diagram with a coordinate system chosen, and use this to find out where Marcus can be in order to hear the phone when it rings.

A box is twice as high as it is wide and three times as long as it is wide. It just fits into a sphere of radius 3 feet. What is the width of the box?

- *Argumentation and Proof:* Mathematical induction and formal proof. Attention should be paid to the distinction between plausible, informal reasoning and complete, rigorous demonstration.

Select any odd number, then square it, and then subtract one. Must the result always be divisible by 4? Must the result always be divisible by 8? Must the result always be divisible by 16? Write convincing arguments or give counterexamples.

The midpoints of a quadrilateral are connected to form a new quadrilateral. Prove that the new quadrilateral must be a parallelogram.

In case the first quadrilateral is a rectangle, what special kind of parallelogram must the new quadrilateral be? Explain why your answer is correct for any rectangle.

Part 4

Desirable areas of focus for students in quantitative majors

What follows is a brief summary of some of the mathematical subjects that are a **desirable** part of the mathematical experiences for students who enter higher education with the possibility of pursuing quantitative majors. No curriculum would include study in all of these areas, as that would certainly be at the expense of opportunities for deep explorations in selected areas. But these areas each provide excellent contexts for the approaches to teaching suggested in Section 1. The emphasis here is on enrichment and on opportunities for student inquiry.

- *Vectors and Matrices:* Vectors in the plane; complex numbers and their arithmetic; vectors in space; dot and cross product, matrix operations and applications.
- *Probability and Statistics:* Continuous distributions; binomial distributions; fitting data with curves; regression; correlation; sampling.
- *Conic Sections:* Representations as plane sections of a cone; focus-directrix properties; reflective properties.
- *Non-Euclidean Geometry:* History of the attempts to prove Euclid's parallel postulate; equivalent forms of the parallel postulate; models in a circle or sphere; seven-point geometry.
- *Calculus**

*Students should take calculus only if they have demonstrated a mastery of algebra, geometry, trigonometry, and coordinate geometry. Their calculus course should be treated as a college level course and should prepare them to take one of the College Board's Advanced Placement Examinations. A joint statement from the Mathematical Association of America and the National Council of Teachers of Mathematics concerning calculus in secondary schools is included as Appendix B.

Comments on Implementation

Students who are ready to succeed in college will have become prepared throughout their primary and secondary education, not just in their college preparatory high school classes. Concept and skill development in the high school curriculum should be a deliberately coordinated extension of the elementary and middle school curriculum. This will require some changes, and some flexibility, in the planning and delivery of curriculum, especially in the first three years of college preparatory mathematics. For example, student understanding of probability and data analysis will be based on experiences that began when they began school, where they became accustomed to performing experiments, collecting data, and presenting the data. This is a more substantial and more intuitive understanding of probability and data analysis than one based solely on an axiomatic development of probability functions on a sample space, for example. It must be noted that inclusion of more study of data analysis in the first three years of the college preparatory curriculum, although an extension of the K-8 curriculum, will be at the expense of some other topics. The general direction, away from a broad but shallow coverage of algebra and geometry topics, should allow opportunities for this.

Appendix A

What follows is a collection of skills that students must routinely exercise without hesitation in order to be prepared for college work. These are intended as indicators—students who have difficulty with many of these skills are significantly disadvantaged and are apt to require remediation in order to succeed in college courses. This list is not exhaustive of the basic skills. This is also not a list of skills that are sufficient to ensure success in college mathematical endeavors.

The absence of errors in student work is not the litmus test for mathematical preparation. Many capable students will make occasional errors in performing the skills listed below, but they should be in the habit of checking their work and thus readily recognize these mistakes, and should easily access their understanding of the mathematics in order to correct them.

1. Perform arithmetic with signed numbers, including fractions and percentages.
2. Combine like terms in algebraic expressions.
3. Use the distributive law for monomials and binomials.
4. Factor monomials out of algebraic expressions.
5. Solve linear equations of one variable.
6. Solve quadratic equations of one variable.
7. Apply laws of exponents.
8. Plot points that are on the graph of a function.
9. Given the measures of two angles in a triangle, find the measure of the third.
10. Find areas of right triangles.
11. Find and use ratios from similar triangles.
12. Given the lengths of two sides of a right triangle, find the length of the third side.

Appendix B

Calculus in the Secondary School

To: Secondary School Mathematics Teachers
From: The Mathematical Association of America
The National Council of Teachers of Mathematics
Date: September 1986
Re: Calculus in the Secondary School

Dear Colleagues:

A single variable calculus course is now well established in the 12th grade at many secondary schools, and the number of students enrolling is increasing substantially each year. In this letter we would like to discuss two problems that have emerged.

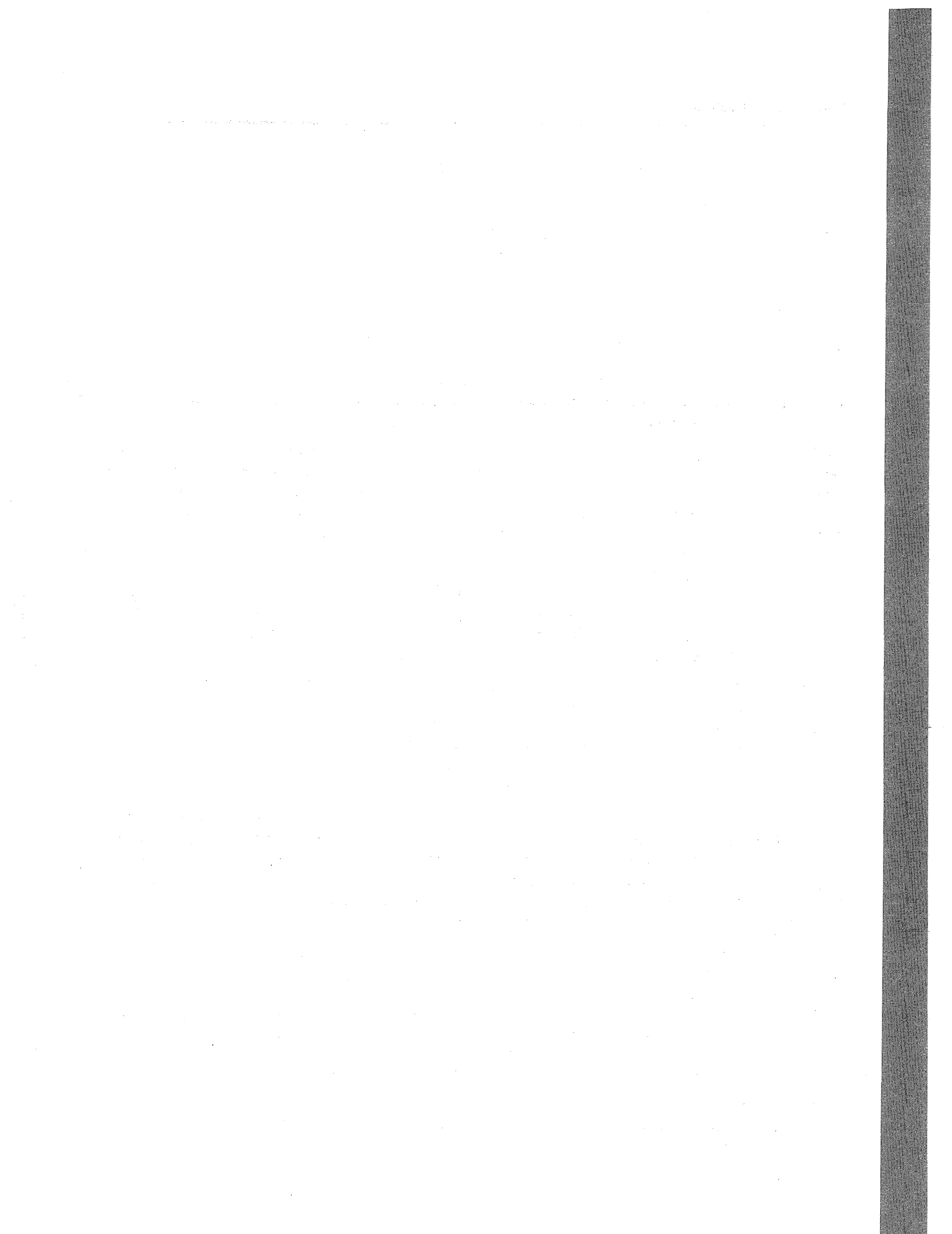
The first problem concerns the relationship between the calculus course offered in high school and the succeeding calculus courses in college. *The Mathematical Association of America (MAA) and the National Council of Teachers of Mathematics (NCTM) recommend that the calculus course offered in the 12th grade should be treated as a college-level course.* The expectation should be that a substantial majority of the students taking the course will master the material and will not then repeat the subject upon entrance to college. Too many students now view their 12th grade calculus course as an introduction to calculus with the expectation of repeating the material in college. This causes an undesirable attitude on the part of the student both in secondary school and in college. In secondary school all too often a student may feel "I don't have to master this material now, because I can repeat it later;" and in college, "I don't have to study this subject too seriously, because I have already seen most of the ideas." Such students typically have considerable difficulty later on as they proceed further into the subject matter.

MAA and NCTM recommend that all students taking calculus in secondary school who are performing satisfactorily in the course should expect to place out of the comparable college calculus course. Therefore, to verify appropriate placement upon entrance to college, students should either take one of the Advanced Placement (AP) Calculus Examinations of the College Board, or take a locally administered college placement examination in calculus. Satisfactory performance on an AP examination carries with it college credit at most universities.

A second problem concerns preparation for the calculus course. *MAA and NCTM recommend that students who enroll in a calculus course in secondary school should have demonstrated mastery of algebra, geometry, trigonometry, and coordinate geometry.* This means that students should have at least four full years of mathematical preparation beginning with the first course in algebra. The advanced topics in algebra, trigonometry, analytic geometry, complex numbers, and elementary functions studied in depth during the fourth year of preparation are critically important for students' later courses in mathematics.

It is important to note that at present many well-prepared students take calculus in the 12th grade, place out of the comparable course in college, and do well in succeeding college courses. Currently the two most common methods for preparing students for a college-level calculus course in the 12th grade are to begin the first algebra course in the 8th grade or to require students to take second year algebra and geometry concurrently. Students beginning with algebra in the 9th grade who take only one mathematics course each year in secondary school should not expect to take calculus in the 12th grade. Instead, they should use the 12th grade to prepare themselves fully for calculus as freshmen in college.

We offer these recommendations in an attempt to strengthen the calculus program in secondary schools. They are not meant to discourage the teaching of college-level calculus in the 12th grade to strongly prepared students.



**MATHEMATICS STANDARDS FOR CALIFORNIA
HIGH SCHOOL GRADUATES**

**CALIFORNIA EDUCATION ROUND TABLE
TASK FORCE ON MATHEMATICS GRADUATION STANDARDS**

February 1997

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California Education Round Table

560 J Street, Suite 390
Sacramento, CA 95814

February 1997

Dear Colleague:

Many Californians are concerned that high school graduates are ill-prepared to enter the workforce or to begin college-level work because they have not mastered high school subjects. In fact, large numbers of high school graduates are unable to express concepts clearly in writing or to use the mathematics necessary to enter many of the technical careers which are available in today's world of work. One key element in meeting the challenge of improving high school students' academic preparation is to make clear what is expected of them by the time they complete their high school careers.

Over the past year, two task forces sponsored by the California Education Round Table and representative of high school teachers, administrators, college and university faculty, and community and business representatives, have worked to develop content standards in English and mathematics that all high school graduates in California should be able to meet in order to succeed in their work careers or in postsecondary education.

In both California and nationally, a lack of agreement on realistic content standards has caused many schools to continue to focus on the number of courses that students are required to complete, or "seat time," for high school graduation rather than on the expected content to be taught and mastered by students. The Round Table content standards are unique in that they reflect the best thinking of K-12, community, and higher education representatives, and have the endorsement of the leadership of all levels of education. In addition, statewide support for the recommended standards from the educational community, California's business leaders and parents' groups has been most positive.

The content standards, first published in draft form in September 1996, were the subject of seven statewide public hearings and were distributed widely to schools, school boards, and interested education organizations, including the academic senates of the public segments of higher education. Comments received during the review process were carefully considered by the task forces, and revisions and additions were made accordingly. **The California Education Round Table has endorsed unanimously the final set of content standards in English and mathematics for high school graduates and urges their implementation in schools throughout the state.**

Each task force has prepared introductory material to their respective subjects that is essential to an understanding of the standards' purpose, their relationship to the high school curriculum, and to preparation for college and employment. The recommended English content standards cover six topics: Reading; Writing; Grammar, Conventions and Usage; Speaking and Listening; Literature; and Using Information. Each content standard describes what students should know, understand, and be able to do to meet the standard and includes examples of individual, small group, or whole

class activities that would, in part, produce evidence of meeting the content standards. A compendium of annotated samples of student work illustrates achievement of the content standards, shows which standards the samples meet, identifies the samples' strengths and limitations, and makes suggestions for improvement.

The mathematics standards for high school graduation are also organized into six topics: Number Sense; Symbols and Algebra; Measurement and Geometry; Functions; Data Analysis; and Mathematical Reasoning. Each topic contains a brief general statement, an explanation of the importance of the topic, the standards themselves, and exemplars. The exemplars are included to help clarify the standards through specific examples of tasks that students should be able to undertake and complete successfully upon graduation. They were written to help make the standards clearer for teachers and are not intended to be test questions.

The Round Table members are aware of many cases in which a more demanding curriculum and clearer expectations have led to increased student performance. Although the content standards were designed to apply to the large majority of students, meeting these standards will be a challenge for many students. Similarly, providing the curriculum required to teach to the standards will be a challenge for many schools. Nevertheless, these standards, though challenging for all, are attainable. The only appropriate exception to mastery of these standards will be some students in designated Special Education programs. The measurement and assessment of student performance with respect to these standards is being addressed by two new Task Forces on Assessment in English and mathematics. This effort, also sponsored by the Round Table, is now underway with an expected completion in spring 1998.

We, as members of the California Education Round Table, urge elected and appointed school officials to consider closely these standards for adoption as benchmarks for the high school curriculum in English and mathematics. As we make this request we reaffirm our commitment to supporting in every way possible the curricular changes necessary in the schools, the professional development required by teachers, and the changes necessary in teacher preparation programs.

Warren H. Fox, Executive Director
California Postsecondary Education Commission

Richard Atkinson, President
University of California

Delaine Eastin, State Superintendent
of Public Instruction
California Department of Education

Nancy Bekavac, President, Scripps College,
Chairman, Executive Committee, Association of
Independent California Colleges and
Universities

Tom Nussbaum, Chancellor
California Community Colleges

Barry Munitz, Chancellor
California State University

California Education Round Table Mathematics Standards for California High School Graduates

INTRODUCTION

Over the last decade, each segment of California education has faced profound policy questions that have at their root the compelling need to improve the quality of education for all students at all levels. Confidence in the public school system is being severely tested by the low performance of California students on standardized examinations, particularly in reading, writing, and mathematics. In response to this concern, Californians overwhelmingly support setting higher goals for students, and are convinced that higher standards will lead to increased student learning. Further, they believe there should be specific guidelines for what students should know and be able to do.¹

The K–12 segment is responding by focusing greater attention on increasing student learning so that all students have increased opportunities to pursue postsecondary education or a career upon graduating from high school. This improvement of academic preparation of high school graduates is central to the California State University’s efforts to reduce the need for remedial education in the CSU system; the University of California’s need for an expanded, diverse pool of fully prepared high school graduates; and the California community college system’s goal of preparing more students for transfer to four year institutions and for employment in technical fields. It is clear that each education segment’s ability to fulfill its mission depends greatly on the success of the other segments. There is an unequivocal understanding that all levels of education, kindergarten through college, must work together. As the Education Round Table recently stated:

“Not only are resources and capacities stretched by competing interests and priorities, but also the problems themselves are inextricably tied to common interests and responsibilities. Now more than ever, we must plan and work together in integrated, focused ways to ensure an acceptable level of academic success for all students, thereby providing equal access to opportunities for higher education, meaningful employment, and full participation in our economy and democratic society.”²

It is within that context and with that spirit of cooperation that the Mathematics Graduation Standards Task Force began its work. In this section, we outline the recent history and charge to the task force, discuss standards generally, and deal with the important differences between these standards and what students are expected to know and be able to do to succeed in college. We close this section by describing the characteristics of high school students who have mastered the content standards that are set forth in section two of the report. Section three focuses on important implementation issues.

A Call to Action

In 1995, California State Superintendent of Public Instruction Delaine Eastin appointed a broad-based task force to make recommendations to improve the quality of mathematics performance in California's schools. Its recent report, "A Call to Action: Improving Mathematics Achievement for All California Students," recommended the immediate adoption of clear and specific content and performance standards for mathematics. Specifically, the panel recommended that graduation standards be adopted "to ensure that standards for high schools reflect what students are expected to know and be able to do as a condition of receiving a high school diploma."³ The report further called for "a balanced program in content, which includes basic skills, conceptual understanding, and problem solving involving a variety of strategies learned from direct instruction and exploration."⁴

Mathematics Graduation Standards Task Force

In 1996, the California Education Round Table⁵ appointed the Mathematics Graduation Standards Task Force, made up of a broad cross-section of Californians, including classroom teachers, school and district administrators, university and college faculty, and community and business leaders concerned with mathematics learning achievement in California schools. Our charge was to agree upon clear and specific mathematics content standards for high school graduation.

The focus of the Mathematics Graduation Standards Task Force is on the knowledge and skills students are able to display upon graduation from high school. However, the task force realizes that sound preparation from elementary through middle school is essential if students are to take the high school mathematics courses necessary to meet the rigorous graduation standards set forth in this document. Standards agreed upon by the task force are important in two contexts. First, they will contribute to the standards-setting process described in AB 265, which calls for the establishment of a Commission on Academic Content and Performance Standards. The Commission is required to first address reading, writing and mathematics at all grade levels 1–12, to be followed by other subject matter areas. The results of the work of the Mathematics Graduation Standards Task Force will be submitted to the Commission and are expected to have significant impact on the Commission's recommendations concerning grade 9–12 standards. The Commission is then to provide recommendations to the State Board of Education for adoption.

Secondly, the task force will contribute to building a common understanding among school teachers, college and university professors, school board members, administrators, employers, and parents about the math-

The focus of the Mathematics Graduation Standards Task Force is on the knowledge and skills students are able to display upon graduation from high school.

The goal of the standard setting process is to encourage higher achievement—not just higher standards.

ematics skills, competencies, and knowledge that students need to acquire in high school in order to graduate and successfully progress to the next level of their lives, whether it be directly into a career, further career preparation, or postsecondary education.

At the outset, it is important to note that the goal of the standard setting process is to encourage higher achievement—not just higher standards. Unless substantial numbers of students *meet* the standards, the overall objective, which is to broaden the opportunities available to them, will not be reached. Achieving a high level of mathematics proficiency will open opportunities to students who wish to pursue jobs in highly technical fields, and to those who wish to pursue a wider array of postsecondary opportunities.

Content Standards and Performance Standards

In addressing the charge from the Round Table to agree upon standards for high school–level mathematics, the task force has been guided by the proposition that content standards must be connected to performance standards. One is meaningless without the other. Content standards define “what” teachers are expected to teach, and what students are expected to learn. Performance standards define the degrees to which students do learn, with higher degrees of mastery expected, for example, of students planning to be majors in science and mathematics in universities than those for high school graduates generally.⁶ In addition, an increasing number of Californians will opt for vocational-technical training. The curricula of these programs are rapidly increasing in complexity and difficulty as careers become more technical. Therefore, candidates for entry and ultimate success in these programs must complete rigorous mathematical coursework at or above baseline performance standards. The content standards contained in this document are intended to be a part of a strong foundation for all students, irrespective of their ultimate career goals or aspirations.

Balanced Standards

Throughout our deliberations, the task force has grappled with the appropriateness of each standard. If we raise the standard unrealistically high, the end result could be more student failure, and teacher and parent discouragement. If we don’t raise the standard high enough, the end result will be the continued failure to prepare our students adequately for their futures. We have sought to achieve an appropriate balance, arriving at high, but attainable standards. Our guiding question has been: What is absolutely essential for a mathematically literate high school graduate to know and be able to do?

On each of the standards we have sought consensus across the broad cross-section of views entertained by task force members. We have concluded that the standards suggested here represent a marked improvement over existing standards. Currently, standards are adopted on a district-by-district level and represent a much lower degree of achievement—much closer, in most districts, to an eighth grade (or even lower) level of proficiency.

Curriculum Options

These standards define what students should learn in order to graduate from high school; the standards do not define a particular sequence of courses. Schools should use these standards to reexamine course content and sequences. Some schools may decide to enroll most students in a common two or three year sequence to complete achievement of these standards. Many students would go on in subsequent courses to complete study that meets expectations for entering college freshmen. Other schools will offer options to students for courses that differ in rigor and pace, but have the achievement of these standards in common. In any case, all students will have open longer the option of preparing for college; and all students will better be prepared for a future increasingly impacted by technology.

Graduation Standards and College-Bound Students

Another important charge to the task force from the Education Round Table, was to “clarify the relationship between these graduation standards and expected competencies for entering freshmen.” The Intersegmental Council of Academic Senates (ICAS) is in the process of adopting a revised “Statement of Competencies in Mathematics Expected of Entering College Students,” which spells out what students need to know and be able to do to be successful in college. That document appropriately sets forth expectations for college-bound students. This document is intended to convey mathematics expectations for *all* students.

The difference between what all high school graduates are currently expected to know and be able to do in mathematics, and what is expected for postsecondary-bound students is enormous. The standards presented in this report move the expectations for all high school graduates substantially closer to the expectations for college-bound students. There remains, however, a gap between the two, and it is important to articulate what the differences are. First, students who expect to enter a four year higher education institution are expected to have completed at least three years of rigorous college preparatory mathematics instruction. Indeed, most students who are admitted to highly selective institutions have completed four or more years of college preparatory mathematics. These standards require a rigorous course of study incorporating core

**What is
absolutely
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mathematically
literate high
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able to do?**

concepts and skills from algebra I and geometry, and include data analysis. Courses designed to prepare students to meet these standards are appropriate for students who intend to continue to study mathematics in preparation for college, as well as students who choose not to study mathematics beyond the content specified in the standards.

Secondly, a higher *level* of performance is expected of college-bound students. All college-bound students are expected to have greater facility and fluency with the basic techniques of mathematics, a deeper understanding of the underlying concepts and the logical reasoning that is central to mathematics. College-bound students should have the ability to solve more sophisticated problems and to use mathematics in a greater variety of applications.

Student Characteristics

Standards are proposed which the task force believes will help develop the following characteristics: High school graduates should have acquired a fluency in fundamental mathematical skills and their use, and realize that they can continue to develop and perfect new skills throughout their study of mathematics at all grade levels. Graduates should also attain the perspective that mathematics provides a way of understanding, with its own structure, and is not just a sequence of algorithms or manipulations of symbols to be used without reflection. Graduates should see mathematics as a way of helping them to understand the world in which they live. Graduates should be able to solve problems that arise both in concrete and abstract situations, including problems that require time and thought, and that go beyond mimicking familiar examples. Graduates should be able to recognize patterns, make conjectures, test these conjectures, and should understand that mathematical assertions require justification based on persuasive arguments.

Adoption of these standards will have enormous implications for all of education, but especially for high schools. In the next section we examine key issues related to implementation.

The activities and behaviors that can accompany the learning of mathematics must not become goals in themselves—understanding of mathematics is always the goal.

IMPLEMENTATION

In this section we explore the impact that adoption of these standards will have on high school mathematics instruction, and on the use of technology in schools—both important implementation issues. In addition, we discuss the kinds of systemic commitment required to make these new standards effective.

Mathematics Instruction

As the Intersegmental Council of the Academic Senates concluded in their statement of Competencies:

“There is no best approach to teaching, not even an approach that is effective for all students, or for all instructors. One criterion that should be used in evaluating approaches to teaching mathematics is the extent to which they lead to the development in the student of the dispositions, concepts, and skills that are crucial to success. It should be remembered that in the mathematics classroom, time spent focused on mathematics is crucial. The activities and behaviors that can accompany the learning of mathematics must not become goals in themselves—understanding of mathematics is always the goal.”⁷

The report identifies the kinds of activities that should be promoted in the classroom. We include them here in summary form:

- Becoming Fluent in Mathematics - Effective classes will provide experiences that lead to students’ acquisition of facility with the basic techniques of mathematics. There are certain necessary skills that students will need to be able to call upon without hesitation.
- Modeling Mathematical Thinking - Effective classes will reflect the enthusiasm that comes in learning with a teacher and others who get excited about mathematics, who work as a team, who experiment and form conjectures.
- Solving Problems - Effective classes will reflect the notion that problem solving is best conveyed by giving students appropriate experience in solving unfamiliar problems, by then engaging them in discussion of their various attempts at solutions, and by reflecting on these processes.

-
- Developing Analytic Ability and Logic - Effective teachers will emphasize a thorough understanding of the subject matter and the development of logical reasoning. A classroom full of discourse and interaction that focuses on reasoning is a classroom in which analytic ability and logic are being developed.
 - Experiencing Mathematics in Depth - Students must delve deeply into well-chosen areas of mathematics. A shallow exploration of a broad range of topics will not contribute to the ability to understand and independently use mathematics.
 - Appreciating the Beauty and Fascination of Mathematics - Effective teachers must nurture the appreciation for the inherent beauty of mathematics.
 - Building Confidence - Effective teachers must create situations in which students are rewarded for being inquisitive, for experimenting, for taking risks, and for persisting in finding solutions they fully understand. This will help students generate confidence in their mathematical ability.
 - Communicating - Effective classes will reflect a student's ability to explain with confidence not only the answer to the problem but the process by which the problem was solved. Students need extensive experience in oral and written communication with extensive feedback in order to develop these skills.

While the development of these dispositions in the student is important, the larger goal is always student understanding of mathematics. As a student develops a mathematical skill, be it a basic skill or a more advanced skill, the development of an understanding is crucial in order that the skill be lasting and one that the student can apply in different contexts. As the ICAS document states:

“Then, as they use the skill in different contexts, they gradually wean themselves from thinking about it deeply each time, until its application becomes routine. But their understanding of the mathematics is the map they use whenever they become disoriented in this process. The process of applying skills in varying and increasingly complex applications is one of the ways that students not only sharpen their skills, but also reinforce and strengthen their understanding. Thus, in the best of mathematical environments, there is no dichotomy between gaining skills

and gaining understanding. A curriculum that is based on depth and problem solving can be quite effective in this regard provided that it focuses on appropriate areas of mathematics."⁸

Use of Technology

Just as there is no "best" approach to teaching mathematics, there is no fail-safe rule which tells us whether or when it is appropriate to use technology to convey the important concepts contained in the standards. Technology is not a substitute for mastery. Neither is it a panacea for the shortcomings of mathematics instruction. It is a tool which, when used appropriately, can enhance mathematics instruction. In making judgments about its appropriateness, teachers and learners must always keep sight of the goal—understanding mathematics.

Educators should actively explore ways that technology can support students in their progress toward achieving understanding of the basic ideas, skills, and techniques represented by these mathematics standards. Many jobs of the future will require the creative use of technology to explore data, analyze information, and solve problems. Opportunities to use technology can assist students in learning basic skills, modeling mathematical thinking, developing analytical ability and logic, and in communicating their understanding of important mathematics concepts. "Technology should not be used just because it is appealing. But it must be used when it can enhance the teaching and learning of mathematics."⁹

The roles of calculators and computers in learning, creating, and applying mathematics are changing almost as quickly as computational technology is advancing. In helping students to learn mathematics, care must be taken in using emerging technologies. Appropriate use might provide students with more access to deeper understanding but caution is needed to prevent rushing to technology in ways that deny students the opportunity to acquire certain skills that can assist their understanding of mathematical concepts and structures. For example, students must be given ample opportunities to master arithmetic computation with whole numbers and fractions without calculators, both in order to be able to perform quick computations with a minimum of effort and in order to develop an internal sense of the meaning and results of such computations through their own experience. On the other hand, technology allows us to perform many more computations with great accuracy and so should be available to students to allow study of realistic data without the burden of performing many time-consuming calculations.

Students will be motivated to learn and mathematics will come alive if examples and applications are available throughout the curriculum.

Opportunity to Succeed—a Shared Responsibility

Increasingly, reformers have come to recognize that merely raising standards is not enough to raise student achievement. Improved curriculum, better prepared teachers, and changes in the organization and management of schools are also necessary to bring about improvement in student performance. The teaching and learning of mathematics should not be limited to formal classes in mathematics. Students will be motivated to learn and mathematics will come alive if examples and applications are available throughout the curriculum. The physical sciences have been the traditional source for applications in mathematics. But virtually every course in the high school curriculum provides opportunities to use mathematics. The social sciences are especially rich in examples in the areas of data analysis and mathematical reasoning. Newspapers and news magazines are readily accessible sources for applications.

Ensuring that students are able to meet these higher standards will require a substantial commitment by all those associated with education. State leadership must provide clear, unequivocal expectations of what students are expected to know and be able to do. Schools must engage parents and the community to help students reach the new standards. Parents need to know and understand early on, even as early as the primary grades, the increased expectations required to attain a high school diploma. Parents should become familiar with the school's mathematics curriculum, engage in a continual dialogue with school personnel regarding their child's achievement levels in mathematics, and provide continuous support and encouragement in the area of mathematics. Students must be prepared to work harder—they must accept the responsibility for their own learning. "Students should realize that their minds are their most important mathematical resource, and that teachers and other students can help them to learn but can't learn for them."¹⁰ Prospective employers and postsecondary institutions must take into account more than grades or receipt of a diploma in assessing readiness for employment or collegiate studies. Mathematics departments and schools of education must collaborate in order to increase their emphasis on the strong mathematical preparation of prospective teachers. Furthermore, teacher preparation programs must expect new teachers to understand the importance of these mathematics content standards, to rethink their role as effective teachers, and to change their teaching practices so that all their students meet these standards. School districts must demand that prospective new teachers have the necessary mathematics background to teach adequately to the new standards. In turn, communities need to hold schools accountable for providing the educational environment that will foster student success.

Ensuring that students are able to meet these higher standards will require a substantial commitment by all those associated with education.

Conclusion

These mathematics standards for high school graduates have been prepared by task force members, motivated by a spirit of cooperation and consensus rarely experienced among representatives of the different segments of education. Our work has been enriched by thoughtful and sound contributions from parents, employers, and community leaders. We have taken our charge from the California Education Round Table most seriously because we believe that changes must occur in the teaching and learning process if the young people of today and tomorrow are to have the skills necessary to compete successfully and prosper in a world and in an economy that is directed by the power of information rather than simply industrial might.

We urge school boards, state and local policy-makers, employers, parents, and the general public to consider these standards closely and to see that they are reflected in the curricula of their local schools. Further, we strongly support university and college efforts to work more closely with the schools in accomplishing the changes necessary to implement these content standards rapidly and pervasively. We must move forward together with purpose and dedication.

References

- ¹ "Californians' Views on Education: Results of the 1996 PACE Poll," Policy Analysis for California Education, Berkeley, CA, April 1996.
- ² "Joint Statement on Collaborative Initiatives to Improve Student Learning and Academic Performance, Kindergarten through College," Education Round Table
- ³ "A Call to Action: Improving Mathematics for All California Students," California Department of Education, Sacramento, CA, 1995, page 5.
- ⁴ Ibid., page 6
- ⁵ The California Education Round Table is a voluntary organization consisting of the Superintendent of Public Instruction, the President of the University of California, the Chancellor of the California State University, the Chancellor of the California Community Colleges, a representative of California Independent Colleges and Universities and the Executive Director of the California Postsecondary Education Commission.
- ⁶ In the section on standards, we take a first step in setting those performance standards by citing exemplars which suggest approaches for developing performance standards based upon these content standards.
- ⁷ "Statement on Competencies in Mathematics Expected of Entering College Students," Intersegmental Council of Academic Senates, Final Draft, 1996, page 3.
- ⁸ Ibid., page 5.
- ⁹ Mathematics Sciences Education Board of the National Resource Council, Reshaping School Mathematics: A Philosophy and Framework of Curriculum, 1990, page 37.
- ¹⁰ "Statement on Competencies in Mathematics Expected of Entering College Students," page 2.

MATHEMATICS STANDARDS FOR CALIFORNIA HIGH SCHOOL GRADUATES

The mathematics standards for high school graduates are organized into six categories: Number Sense; Symbols and Algebra; Measurement and Geometry; Functions; Data Analysis; and Mathematical Reasoning. Each category contains a brief general statement, an explanation of the importance of the category, the standards themselves, and exemplars. The exemplars are included to help clarify the standards through specific examples of tasks that students should be able to undertake and to complete successfully upon graduation. The exemplars are not intended to be used as test questions themselves. They were written particularly to help make the standards clearer for teachers. They might also suggest approaches for developing performance standards based upon these content standards.

NUMBER SENSE

People and societies first encounter mathematics through numbers. There are many kinds of numbers, including counting numbers, negative numbers, fractions, and decimals. For the high school graduate, number sense goes beyond the simple ability to calculate with numbers. It includes the ability to estimate, to represent numerically, and to judge the reasonableness of quantitative results.

Numbers enable us to do more than just count—we measure, compare, and predict using numbers. They enable us to quantify aspects of the universe, of our society, and of our individual lives. To compare sizes, we use ratios or fractions. To record changes over time, we use positive and negative numbers. To measure the physical world, we use decimals. To make predictions, we use all of these kinds of numbers.

Everyone must be able to perform arithmetic with facility, without reliance on a calculator. Calculators can help us to understand and to work with realistic data and thus can motivate and lead to a greater appreciation for numbers and for the power of mathematics. But we must have the numerical sophistication that calculators can't provide that allows us to compare, estimate, understand trends, and analyze quantitative information in order to make reasonable decisions in areas of public policy and personal finance. It is impossible to understand modern technology and finance without a strong sense of numbers. A high school graduate with a deep understanding of numbers also has a foundation for pursuing higher levels of mathematics or any quantitatively based area of inquiry.

Standard I: High school graduates will perform appropriate numerical calculations (including addition, subtraction, multiplication, division, and exponentiation) and draw conclusions from the results. They will convert between different units of measurement.

Even though computers can perform increasingly complex computations with incredible speed, much of what we must do to analyze common situations requires simple arithmetic. High school graduates must maintain the skills that have been expected of them through elementary and middle school, which includes a mastery of the basic skills of arithmetic. Arithmetic enables us to check the reasonableness of many complex calculations.

Exemplars for Number Sense Standard I:

Your neighbor just bought 1,600 square feet of sod to cover her back yard, which is square in shape. How many feet of fencing would she buy in order to enclose the yard on three sides?

The distance from the earth to the moon is about 238,700 miles. There are 5,280 feet in a mile, and a dollar bill is approximately 6 inches long. How many dollar bills would have to be placed end-to-end to reach from the earth to the moon?

There are a little more than two and a half centimeters in an inch (2.54 cm). So there are a little more than 30 cm in a foot. How many inches less than a foot is 30 cm?

NUMBER SENSE (continued)

Standard II: High school graduates will understand that numbers can be represented in different forms and will be able to convert among different representations of numbers as fractions, decimals, percents, and scientific notation.

Numbers occur to us as fractions, in some cases, and as decimals in others. Fair allocation of resources typically presents us with fractions, while using our calculator presents us with decimals or with numbers in scientific notation.

Exemplars for Number Sense Standard II:

In a recent syndicated cartoon, the boss of a company was upset because he learned that 40% of all employee absences occurred on Mondays and Fridays. Explain the mathematics behind the joke.

Express as a fraction the amount by which $\frac{1}{3}$ exceeds 0.333233333...

My calculator computed 11^{12} as 3.13842837672E12. It computed 12^{11} as 743008370688. Which number is larger?

A high density diskette has a capacity of 1.4 MB (1.4×10^6 bytes). How many high density diskettes would you need to match the capacity of a 1.4 GB (1.4×10^9 bytes) hard drive?

By examining the division process, explain why the fraction $\frac{5}{7}$, when written as a decimal, is a repeating decimal. Extend your reasoning to $\frac{9}{17}$.

Standard III: High school graduates will compare sizes of numbers, including very small numbers and very large numbers, in many ways, using differences, ratios and proportions, percentages, and location on the number line.

Whenever we make a choice among options that is based on quantitative information, we compare numbers. We do this in decisions about economics, politics, science, health care, and other areas. The most basic understanding of numbers depends upon comparing them.

Exemplars for Number Sense Standard III:

Which is the better value: 8-ounce cans of tomato sauce that you can buy 3 for a dollar, or 14-ounce cans that cost 60 cents each?

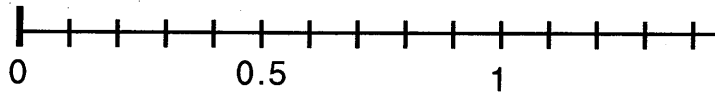
What power of ten best approximates 1,084-108.4?

List the following numbers in order, from smallest to largest: $\frac{3}{9}$, $\frac{3}{8}$, 0.37, 0.38.

Exemplars for Number Sense Standard III (continued):

The price of a book, after it has been marked down 15%, is \$12.75. What was the original price of the book?

On the number line below, locate the approximate position of the number $\frac{1}{9}$.



Standard IV: High school graduates will estimate answers reliably and will check the reasonableness of answers they have calculated. When they use calculators or computers to perform calculations, they will correctly interpret their results.

Whenever we solve a real problem using computations, the resulting number must be interpreted in the context of the problem. Of course it must make sense as a solution to the problem.

Exemplars for Number Sense Standard IV:

You're in charge of ordering buses for the senior outing. Each bus holds 60 passengers, and a total of 381 people will be traveling by bus to the event. How many buses should you order?

Without using a calculator, find the largest whole number that is less than the square root of 43.

Symbols and Algebra

Algebra developed gradually over centuries as a method for representing numbers by symbols or letters, and then for using the rules that numbers obey to generate rules for working with the letters. Algebra then allows the skilled user to solve problems in which we have information about an unknown number from which we must determine the number. The process of abstraction, by which we transform a problem from the natural world into an equation to be solved, is perhaps the most challenging, but also enables us to think more abstractly, to tie together apparently different situations through generalization. Expressing operations from a naturally occurring situation in algebraic terms is also an important practical skill. For example, when using a spreadsheet, we often need to find the formula to describe the calculations we want the computer to make.

We also graph algebraic equations, on a coordinate plane, which allows the equations to be represented geometrically as curves. This connection between algebra and geometry allows for a deeper understanding of both subjects by providing powerful methods for solving algebraic problems by geometric methods and powerful methods for solving geometric problems by algebraic methods. While graphing calculators can enhance our ability and efficiency in this regard, these are the standards that students must be able to meet without their use.

Standard I: High school graduates will use symbols to represent unknown quantities. They will evaluate algebraic expressions and formulas, in both concrete and abstract settings, and correctly interpret the results.

Formulas are increasingly pervasive in the workplace and in personal finances. Correctly evaluating them is the most basic skill in using them.

Exemplars for Symbols and Algebra Standard I:

The company where Tony works has a savings program for its employees. Each year the employee pays an amount P into the savings account. The account earns interest at the rate of $R\%$. The formula for how much money (A) will be in the account after ten years is

$$A = P \cdot \frac{(1 + \frac{R}{100})^{11} - 1}{\frac{R}{100}}$$

How much will Tony have in his account after ten years if he pays \$60 per year into the account, and it earns 10% interest?

Each side of a square has length x . What measurement on the square could $4x$ represent?

Symbols and Algebra (continued)

Standard II: High school graduates will analyze suitable problems that arise in everyday contexts and represent them as equations or inequalities to be solved.

Before algebraic skills can be used to solve a problem, the problem must typically be correctly stated as an algebraic equation to be solved. The person who can make the abstraction to find the equation that describes some situation has a distinct advantage in solving problems.

Exemplars for Symbols and Algebra Standard II:

A club is planning for its financial future. Their annual expenses this year amount to \$2,000, and they expect that this will go up by \$300 each year. Their fundraising efforts bring in \$2,800 this year, and they expect that this will go up by \$100 each year if they don't change their fundraising strategies. Write an equation that relates their annual expenses to the years that pass. Write an equation that relates their income from fundraising to the years that pass.

An electronics store reduced all of their stereo prices by 30%. Then they reduced every stereo price another \$10. Find an equation that relates the original price of a stereo to its price after both price reductions.

Standard III: High school graduates will have a facility with algebraic manipulations so that they can perform simple algebraic computations easily and routinely. They will solve linear equations and inequalities, and will solve two linear equations in two unknowns. They will understand the fundamental concepts of slope and intercept for lines, and use them both algebraically and graphically. They will solve quadratic equations using algebraic techniques and estimate and interpret solutions using graphical techniques. Graduates will correctly use powers and roots of variables, as inverse operations.

Real problems involving an unknown quantity frequently become linear or quadratic equations to be solved.

Exemplars for Symbols and Algebra Standard III:

If the projections made by the club in the exemplar above for income and expenses are correct, how long will it be until the fundraising income just covers their expenses?

A ball is launched straight up into the air at a rate of 64 feet per second. Its height h above the ground (in feet) after t seconds is

$$h=64t-16t^2$$

How high is the ball after 1 second? When is the ball 64 feet high? For what values of t is $h=0$? What events do these represent in the flight of the ball?

Exemplars for Symbols and Algebra Standard III (continued):

If $ad-bc=1$, then $\frac{a}{b} - \frac{c}{d} - \frac{1}{bd} = ?$

If $\sqrt{x^3} = 8$, then what could x be?

To compute the deduction that you can take on your federal tax return for medical expenses, you must deduct 7.5% of your adjusted gross income from your actual medical expenses. If your actual medical expenses are \$1,600 and your deduction is less than \$100, what can you conclude about your adjusted gross income?

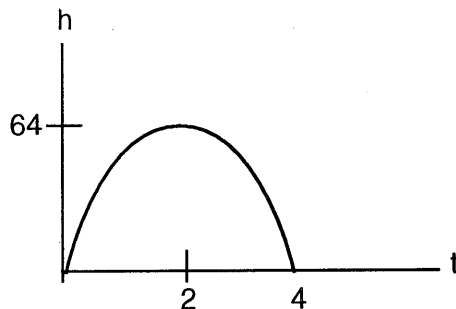
Standard IV: High school graduates will correctly interpret significant features of graphs of algebraic equations, in the context of the situation that gave rise to the equations.

Interpreting the features of a graph is a part of understanding the information that consumers are presented with on a daily basis.

Exemplars for Symbols and Algebra Standard IV:

Graph the equations for the annual expenses and fundraising income for the club mentioned in the exemplar under Standard II. Use these graphs to estimate when the annual expenses and fundraising income will be the same.

A ball is thrown straight up into the air at a rate of 64 feet per second. Its height h above the ground (in feet) after t seconds is shown in the graph below:



What is the significance of the behavior of the graph at the point $(2,64)$?
What is the significance of the behavior of the graph at the point $(4,0)$?
How far has the ball traveled when it hits the ground?

Symbols and Algebra (continued)

Standard V: High school graduates will show why their solutions to simple algebraic problems are correct, coherently explaining the rules they apply to find the solutions.

Whenever conclusions are drawn from known facts, the way in which the conclusion follows from those facts should be understood.

Exemplars for Symbols and Algebra Standard V:

Solve the equation $x^3 - 4x = 0$, explaining each step as you go.

Your class is asked to solve the following equation for x , and to explain your steps:

$$x = \sqrt{8 - x^2}$$

Your friend in this class submits the following solution:

$x = \sqrt{8 - x^2}$ is the same thing as saying $x^2 = (\sqrt{8 - x^2})^2$, since all I did was to square both sides. This means $x^2 = 8 - x^2$.

Add x^2 to both sides to get $2x^2 = 8$. Now divide both sides by 2 to find $x^2 = 4$. Take the square root of both sides to get $x = 2$.

Comment on the errors in this work.

Measurement and Geometry

When we measure, we use geometry. When we build, we use geometry. When we use a map, we use geometry. As societies have developed, they have seen the importance of geometry as they try to understand the motion of the stars, to parcel land, to mark the passage of time, to navigate—whether it be across a sea or to the moon.

Geometry is the mathematics of points, lines, angles, surfaces, and solids. It allows us to describe, categorize, and measure these objects. By doing this, it provides a language to describe and analyze aspects of the world we see around us. Understanding the world requires that we recognize the underlying shapes in the physical world.

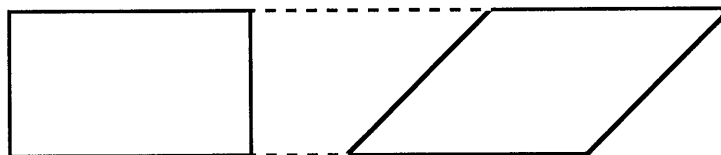
Geometry involves absolute precision, which is an ideal that is only approximated by objects in the natural world but that ties together the similarities of these objects. The exactness of geometry is thus an abstraction, requiring exact arguments. In this way, geometry provides an extraordinary arena for the development of abstract reasoning skills.

Standard I: High school graduates will know how to measure geometric objects. They will also make reasonable estimates of sizes and understand how the measurements of length, area, and volume of various abstract and natural world objects are developed from the basic units of measurement (a unit distance, which leads to a unit square, and then a unit cube). Graduates will use spatial visualization to construct useful models for three-dimensional objects from verbal descriptions, and will solve elementary problems about those objects.

In order to know how big objects are, we need a common standard of comparison. Well-developed notions of distance, area, and volume are crucial to communication about the objects we see. Memorization of formulas, such as the surface area of a cone, the volume of a sphere, or the perimeter of a pentagon is not as useful as the understanding that these measurements follow logically from elementary notions, because this logical development allows us to find measurements for objects we've never seen before.

Exemplars for Measurement and Geometry Standard I:

Use dissection to show that the rectangle and parallelogram shown below (which have the same base) have the same area.



Exemplars for Measurement and Geometry Standard I (continued):

You want to wrap a package in wrapping paper in order to send it across the country. The package is a shoe box that is 12 inches long, 6 inches wide, and 4 inches high. If you are going to use a rectangular piece of paper to wrap this box, then what should the dimensions of the piece of paper be? Find dimensions for another box that has the same volume as this box. Which box would need more wrapping paper?

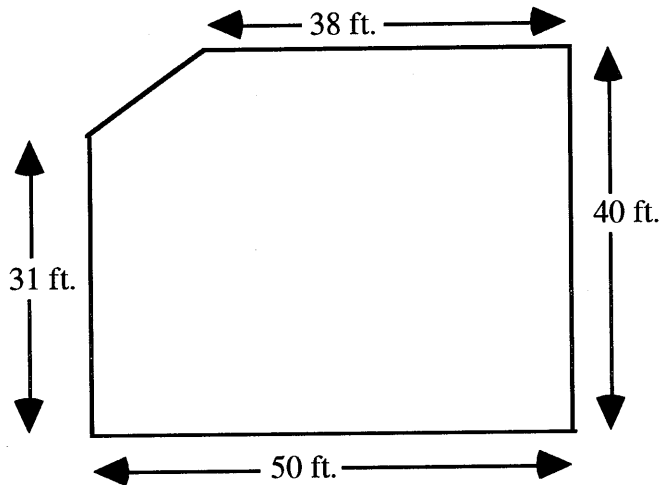
A pyramid is built out of cardboard using 4 equilateral triangles for the sides and a square for the base. The base is 4 inches on each side. If you want to tape each seam on the pyramid, how much tape will you need?

Standard II: High school graduates will know the elementary properties of standard plane geometric objects, including lines, squares, rectangles, triangles, and circles (including, but not limited to: the Pythagorean Theorem, the sum of angles in a triangle, the measure of angles inscribed in a semi-circle, which angles are equal and which are supplementary when a pair of parallel lines is met by a transversal, and the basic trigonometric ratios—sine, cosine, and tangent—for right triangles). They will not only be familiar with the defining characteristics of these families of objects, but also will understand that from basic properties follow other properties, and will understand why.

The elementary properties of standard geometric objects are useful in a myriad of subtle ways in our lives, such as when we build something, navigate, pack objects into tight spaces, or draw with perspective. The deductive reasoning skills that can be honed in the study of geometry serve us in all aspects of our lives in which we make reasoned judgments.

Exemplars for Measurement and Geometry Standard II:

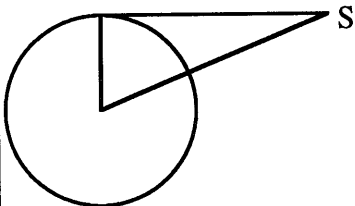
The diagram below shows the overall floor plan for a house. It has right angles at three corners.



What is the area of this house, in square feet?

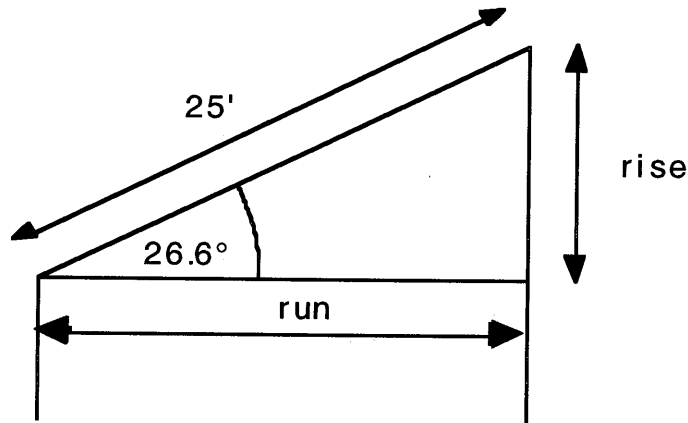
What is the perimeter of this house?

A satellite is 22,300 miles from the nearest point on the Earth, and the radius of the Earth is 3,963 miles. At the time when the satellite just appears to you on the horizon, how far is it from you?

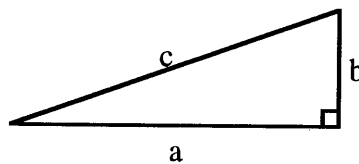


Exemplars for Measurement and Geometry Standard II (continued):

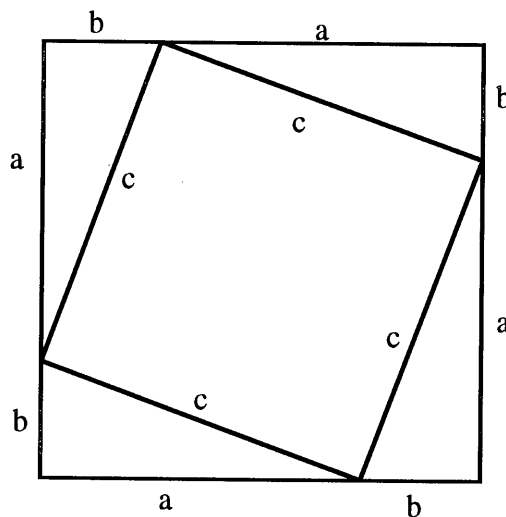
A roof rises at an angle of 26.6° . If the distance to the crest of the roof is 25 feet, what is the rise of the roof? What is the run of the roof? What is the ratio of the rise to the run?



Starting with an arbitrary right triangle



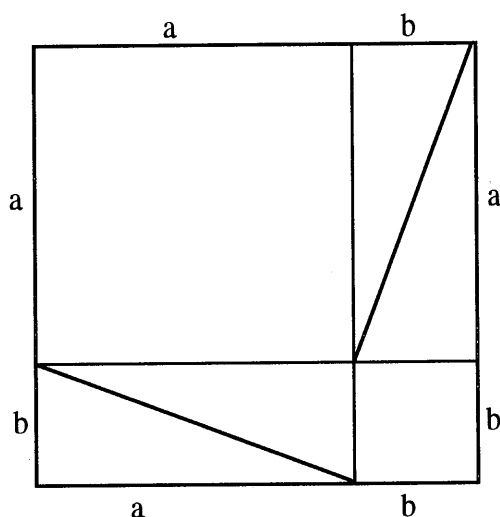
a square whose sides have length $a+b$ can be constructed involving four copies of this triangle as shown:



Show that the inside quadrilateral is a square.

Exemplars for Measurement and Geometry Standard II (continued):

If we slide the four triangles in the square we can make the following figure:



By examining these two figures, explain why the Pythagorean theorem is true.

Standard III: High school graduates will recognize congruent and similar objects, both from the natural world and from abstract definitions, both in two- and three-dimensions. They will know sufficient conditions to determine congruence or similarity. They will use proportional reasoning on similar objects. They will recognize symmetries for two- and three-dimensional objects.

Our understanding of scale models is based upon similarity. Lack of understanding that there are three different notions of what could be meant by saying that one object is twice as large as another (twice the perimeter, twice the area, and twice the volume) can lead to serious errors in purchasing materials, for example.

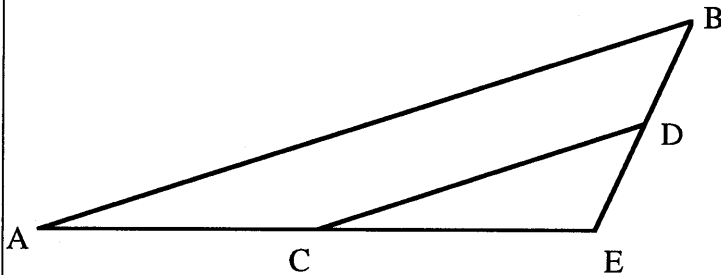
Exemplars for Measurement and Geometry Standard III (continued):

A softball and a baseball are similar in shape and have a similar seam pattern. The circumference of a baseball is 9 inches and the circumference of a softball is 12 inches. The seams on the baseball are 16 inches long. How long would you expect the seams on the softball to be?

To figure out how tall a tree is, I stand in its shadow so that the shadow of the top of the tree coincides with the shadow of the top of my head. I am 6 feet tall, and my shadow is 8 feet long. I am standing 58 feet from the base of the tree. How tall is the tree?

Exemplars for Measurement and Geometry Standard III (continued):

In the figure below, line AB is parallel to line CD. If $\overline{DE} = 1$, $\overline{BE} = 2$, and $\overline{AE} = 5$, then find \overline{CE} .



Standard IV: High school graduates will know how to locate points and measure distances in the coordinate plane. They will set a geometric problem in a coordinate plane, when appropriate, to help solve it.

The coordinate plane (the x-y plane) is the setting for graphical presentations of data, and graphical representations of equations and functions. It is also the bridge between algebra and geometry, which provides for a deeper understanding and more powerful techniques in both subjects.

Exemplars for Measurement and Geometry Standard IV (continued):

Find the distance between the point (0,2) and the point (4,5). Find another point on the y-axis whose distance from (4,5) is this amount.

Two children are standing together in the middle of a flat field. One of them walks 100 feet East, turns left 90 degrees and walks 60 feet North and then stops. The other child walks 140 feet South, turns left 90 degrees and walks 60 feet East, then turns left 90 degrees again and walks 80 feet North and then stops. How far apart are the two children now?

Functions

A function represents the dependence of one quantity on another; it describes how the first quantity changes as the other one changes. For instance, if a ball begins to roll down an incline, the distance it has traveled, say in meters (one quantity), depends on the time that has elapsed, say in seconds (the other quantity). The functional relation allows us to find the distance the ball has rolled if we know the time that has elapsed. The quantities in a functional relation (in the example above, distance and time) can be represented by variables, and the function can be thought of as a relationship between these variables. Functions are tools that allow us to investigate the relationships between things that can be quantified (such as time, distance, dollars, temperature, weight, speed, population, etc.), and thus help us to solve problems.

A function can be represented in several ways. One way is numerically, with a table of values of the two quantities. Another way is symbolically, with a formula that shows how to compute one quantity from the other. A third way is graphically, using a coordinate plane. A fourth way is using words, which can show the connection between a function and a situation it represents.

No matter which incline is used, the function that represents the distance the ball has traveled since starting to roll is always a special kind of function known as quadratic. Quadratic functions also can be used to represent other functional relationships, such as the area of a circle as a function of its diameter or the stopping distance for a car as a function of its speed. This is an example of how families of functions can be a powerful mathematical concept, since we can use what is learned in solving one problem to help solve others.

Some functions are used to describe cyclical phenomena, such as the amount of the moon that is visible from a point of the earth and that is illuminated by the sun, or the height above ground of a seat on a Ferris wheel, as functions of time. For these functions (called periodic), the time it takes to complete a single cycle (a period) and what happens during a period completely determine the function. For example, we only need to know how much of the moon is illuminated throughout one lunar cycle and the length of a lunar cycle in order to determine every time when a quarter of the moon will be illuminated.

Standard I: High school graduates will represent functions in several ways (numerically, symbolically, graphically, and verbally), recognize distinctive characteristics of a function from its several representations, and carry information about a function from one representation to another.

The interplay among different representations of a function provides us with a wealth of perspectives, which can give us unexpected insights into the relationship that the function describes.

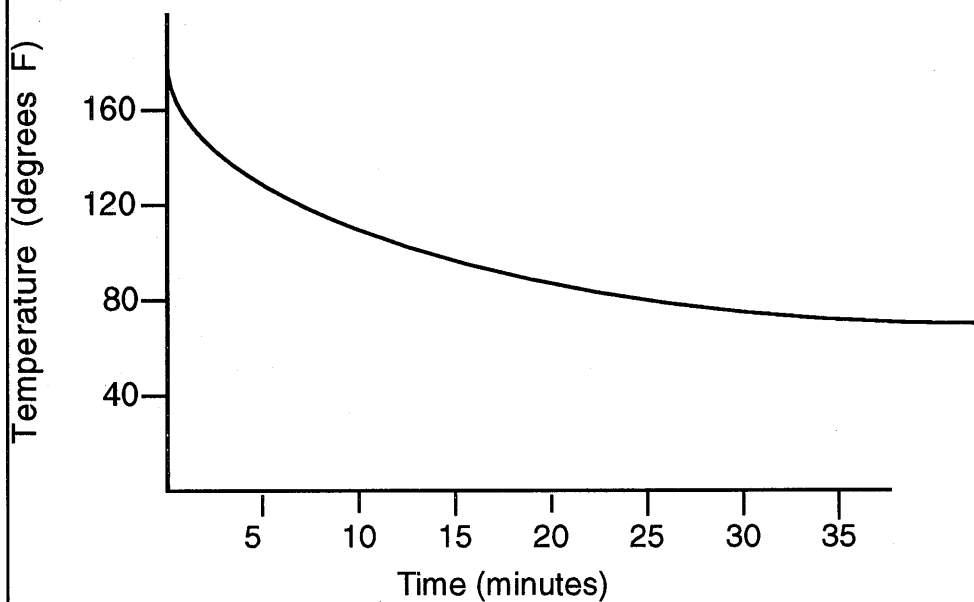
Exemplars for Functions Standard I:

Plot these data:

x	y
-2	-1
1	5
3	9
7	17

on a coordinate plane. Find a plausible and simple graph of a function that passes through these points. Find an algebraic representation for this function.

The following graph shows the temperature of a cup of coffee as a function of the amount of time since it was left in a room.



What was the temperature of the cup of coffee when it was left in the room?

What do you think the temperature of the room was?

Sketch a plausible graph of the temperature of a cup of iced tea as a function of the amount of time after it is left in the same room.

Functions (continued)

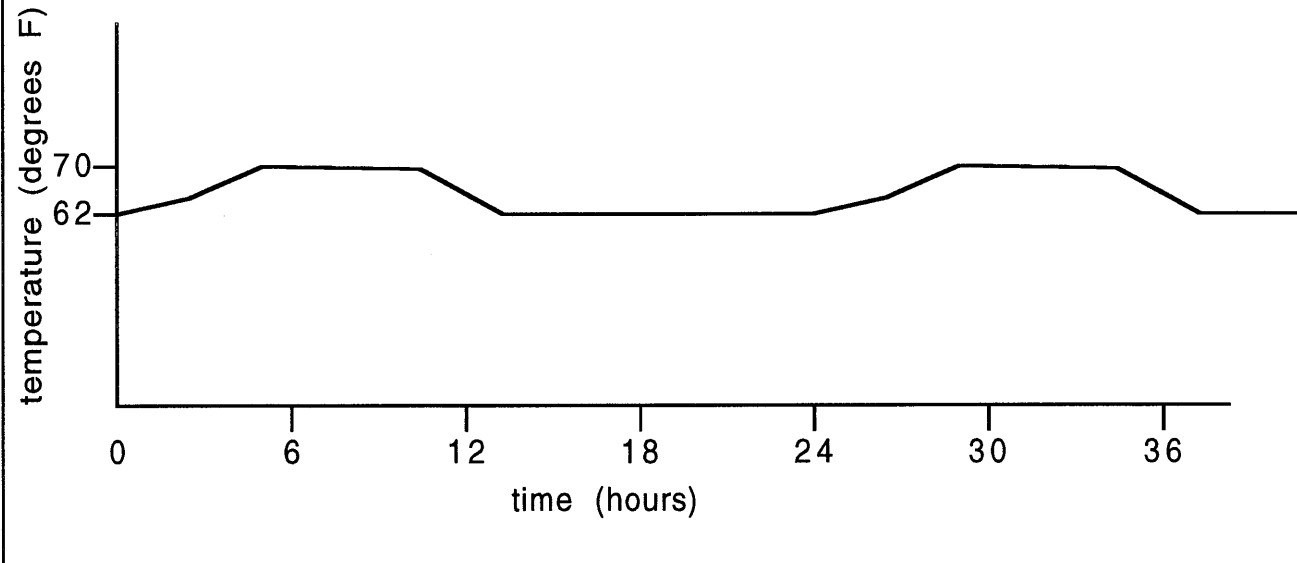
Standard II: High school graduates will recognize and use the basic families of functions: linear, exponential, and quadratic. They will recognize the period of a periodic function from its graph and use the values of the function through one cycle to find its values everywhere.

Many of the functions that graduates will encounter in applications, such as in economic, biological, physical, and sociological contexts, have symbolic representations that are linear, exponential, quadratic, or periodic. These functions are also the most readily computed. Periodic functions pervade the natural world, where understanding the behavior of a function through a single period supplies information about the entire function.

Exemplars for Functions Standard II:

A bacterial colony begins with 5,000 individuals, and doubles in size every day. Find a formula for the function that describes the number of individuals in the colony as a function of time (in days). Use this to determine the size of the colony after 9 days.

The graph below shows the temperature in an office building as a function of time, where the time $t=0$ corresponds to 6 AM on a Monday morning. Find a plausible explanation for the shape of this graph. What do you expect the temperature will be at time $t=102$?



Functions (continued)

Standard III: High school graduates will apply their knowledge of functions to help understand the world around them.

Citizens need to know how to interpret public policy debates that use mathematical models. Workers use mathematical models to manage the technical processes that are becoming more and more common in the workplace; functions are one of the most basic tools used to make these models. Students must put their knowledge of functions to work in situations such as they will encounter outside of school.

Exemplars for Functions Standard III:

Two proposals for income taxation are being considered. With both of the proposals there would be no deductions.

Under the first proposal, the taxes would be 20% of an individual's income.

The second proposal would tax an individual's first \$10,000 of income at 10%. The next \$20,000 would be taxed at 20%, and anything over that amount would be taxed at 30%.

Both of these proposals define the tax owed as a function of income. Sketch graphs of these two functions. Which proposal results in lower taxes for someone with an income of \$20,000? \$50,000? \$100,000?

Data Analysis

As citizens and consumers, each of us is presented with large amounts of data on a regular basis. Often these data have been organized, summarized, or graphed and we must understand this information in order to make informed decisions. Many times conclusions have been drawn from data, and we must decide for ourselves whether the sampling was representative and whether the conclusions follow. An understanding of data analysis is increasingly important for every Californian.

When we have a large amount of data, there is a question of how to organize it, or how to summarize it, so as to make it understandable. For example, if we are looking at a list of all of the heights of sophomore girls at a high school, perhaps a list that includes 400 heights, then it is difficult to understand much by just looking at the list. But if we construct a chart that shows all of the different heights and how many students have each height (a histogram), then we can more readily digest the information. So, one of the most important aspects of data analysis is the graphical organization of data. We might also want to use numbers to describe the data, such as the average height and some measure of how spread out the heights are. These are examples of statistics, another important part of data analysis.

If we keep track of two different variables, such as the heights of the sophomore girls, and the heights of their mothers, then we might keep this information as 400 different pairs of numbers. To organize these data, we might plot these ordered pairs as points (creating a scatter plot), and then be able to see trends such as whether taller sophomore girls tend to have taller mothers. If the points in this scatter plot fall close to a line, then we might use this line to help us see the trend more clearly.

If we are trying to use the information we have gathered from this one high school to make estimates of heights of sophomore girls for the entire district, then we are making inferences. We are now concerned with two main issues: Was the sampling from the one school a fair representation of the entire district? If so, the second issue is that we can't assert that the average height for the district is exactly the same as the average height for the one school, only that it is probably close to that number. These are also important parts of data analysis, sampling from a large population and using the data from the sample to draw conclusions about the entire population.

Data Analysis (continued)

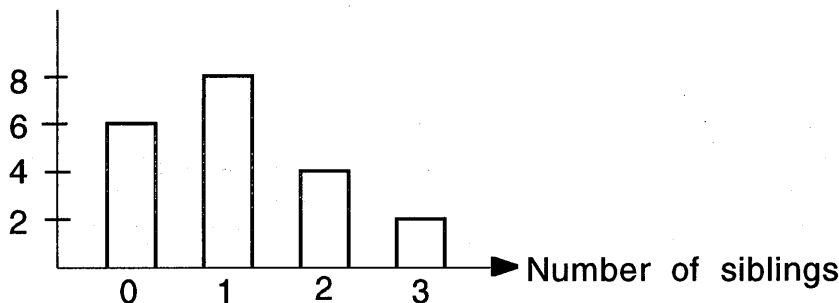
Standard I: High school graduates will organize single-variable data using graphs, such as histograms, and will correctly interpret graphical presentations of single-variable data. They will also correctly compute and interpret measures of central tendency, such as the mean and median, and they will correctly interpret a variety of measures of dispersion, such as range and percentiles.

In the information age, more and more data are compiled, organized, summarized, and presented for our use. Both as citizens and consumers, we must correctly understand these graphs and statistics in order to reach appropriate decisions.

Exemplars for Data Analysis Standard I:

Two algebra classes are given an exam, on which scores of 70-79 are given a grade of C. The two classes have the same mean (72) and the same range (44). Give examples to show that it is possible that in one class most of the students received Cs and in the other class no student received a C.

In a class of 20 students, the students are asked how many siblings (brothers and sisters) they have. The results are shown in this bar chart:



For example, the chart shows that 6 students have no siblings and 8 students have 1 sibling. What is the average number of siblings for students in this class?

A survey of rents is made in two neighborhoods, and reported in the newspaper. The article says that both neighborhoods have an average of \$650 per month for rents, and concludes that the neighborhoods have comparable rental rates. But the article also says that an apartment that rents for \$800 is at the 80th percentile in the first neighborhood but that an apartment that rents for \$800 is at the 90th percentile in the other neighborhood. Write an explanation, in simple terms, of what this information about percentiles means. Also, what do you think about the newspaper's conclusion that the neighborhoods have comparable rental rates?

Data Analysis (continued)

Standard II: High school graduates will organize two-variable data using graphs, such as scatter plots, and will correctly interpret graphical presentations of two-variable data. They will recognize when a scatter plot shows a clear trend, such as when the data points lie near a line, and will use such trends to make predictions.

When we consider statistics concerning a population, we are often interested in how one characteristic of the individuals in the population relates to another. This occurs in finance, social science, health care, natural science, and virtually every area in which statistics are used. We can often use these two-variable techniques to make predictions about how one characteristic might influence another.

Exemplars for Data Analysis Standard II:

The number of miles that a car is driven is often related to the age of the car. In California, the averages are:

Age of car (in years)	0	5	10	15	20	25
Average annual miles driven	15,000	10,800	8,000	6,100	4,100	2,600

What trend do you see in these data? How many miles do you think that a car that is 12 years old is driven, on average? Explain how you found this estimate.

A manufacturer of hard disks has the following price schedule:

capacity (MB)	420	535	850	1080	1280	2,200	4,300
price	\$209	\$235	\$255	\$315	\$345	\$605	\$1015

Make a scatter plot of these data. Draw a curve that reasonably fits the data, and use this to estimate what the price would be for a 1,500 MB hard disk. What is the apparent cost for each additional MB of capacity? How much more capacity could be purchased for an additional \$50?

Standard III: High school graduates will understand the meaning of the probability of an event, and compute the probability of an event that is a collection of equally likely outcomes. They will also compute probabilities of simple combinations of events whose probabilities are known, such as independent events or disjoint events. They will compute and interpret expected values.

Many choices that we make occur in situations where the results are uncertain. To quantify that uncertainty in order to make decisions in our best interest, we need to understand probability.

Exemplars for Data Analysis Standard III:

Arlene and her friend want to buy tickets to an upcoming concert, but tickets are difficult to obtain. Each ticket outlet will have its own lottery, so that everyone who is in line at a particular outlet to buy tickets when they go on sale has an equal chance of purchasing tickets. Arlene goes to a ticket outlet where she estimates that her chance of being able to buy tickets is $\frac{1}{2}$. Her friend goes to another outlet, where Arlene thinks that her chance of being able to buy a ticket is $\frac{1}{3}$.

- a. What is the probability that both Arlene and her friend are able to buy tickets?
- b. What is the probability that neither Arlene nor her friend are able to buy tickets?
- c. What is the probability that at least one of the two friends is able to buy tickets?

Carla has made an investment of \$100. She understands that there is a 50% chance that after a year her investment will have grown to exactly \$150. And there is a 20% chance that she'll double her money in that year, but there is also a 30% chance that she'll lose the entire investment. What is the expected value of her investment after a year?

Standard IV: High school graduates will understand that sampling is a powerful and necessary tool for statistical estimation. They will also understand that variations occur among samples, and thus that while a sample statistic can be a good estimator, it will not in general equal the value for the entire population. They will also understand that unrepresentative samples can lead to misleading results. In considering studies based on samples, they will routinely question biases in the samples.

Many people attempt to influence our decisions by presenting statistics, and by making inferences from those statistics. We must decide for ourselves whether the sample from which the data was drawn is representative and whether it justifies the conclusions.

Exemplars for Data Analysis Standard IV:

Prior to an election, a company polls 500 likely voters at random, and finds that 260 of them plan to vote for the local school bond. Do you think that the polling company is justified in reporting that the school bond is more likely than not to receive a majority of the votes in the election? Explain your reasoning. Do you think that the polling company is justified in reporting that the school bond will receive 52% of the votes in the election? Explain your reasoning.

A record store wants to survey some students at a high school to determine the musical preferences of the student body, to decide what CDs to stock. They poll 94 students who attend a student choir performance, and find that 12 prefer Country/Western music, 15 prefer rock, 32 prefer classical, 25 prefer jazz, and 10 prefer rap music. If the survey is all they have on which to base a decision, then what percentage of their stock should they allocate to classical music? Discuss any possible flaws you see in the methodology of this survey.

Mathematical Reasoning

Mathematics provides an extraordinary opportunity to encounter reasoning in one of its purest forms, to establish mathematical truths with a certainty that is rare in other settings. It gives people a chance to hone reasoning skills that can then be applied in much more ambiguous contexts. The importance of reasoning to mathematics cannot be overstated. Constructing valid arguments and criticizing arguments is part and parcel of doing mathematics. If reasoning is not being developed in the student, then mathematics becomes simply a matter of following set procedures, mimicking examples, without thought to if or why it makes sense.

There are two broad types of mathematical reasoning—inductive and deductive. When we recognize a pattern and formulate a statement that describes it to apply to a new situation, we are reasoning inductively. For example, when we see that the sum of the first 3 odd counting numbers is 9 ($1+3+5=9$), that the sum of the first 4 odd counting numbers is 16 ($1+3+5+7=16$), and that the sum of the first 5 odd counting numbers is 25 ($1+3+5+7+9=25$), we might find a pattern. Reasoning inductively, we would expect that the sum of the first 10 odd counting numbers is 100. This is quite different from the reasoning we'd need to do to know that the pattern we believe we've seen actually holds no matter how many consecutive odd counting numbers we add—this is where deductive reasoning is needed. When we find conclusions that must follow from our hypotheses, and show why this is the case, we are reasoning deductively. While some abstraction is always involved in mathematical reasoning, the reasoning can be in a very concrete or applied setting or can be based upon observations of physical or sociological data.

The certainty of mathematics is part of what makes it both useful and interesting. Powerful systems of notation and conventions of expression have been developed for the deductive reasoning that gives us confidence in the truth of mathematical statements. Understanding mathematical concepts typically involves understanding why mathematical statements are true. When this deductive reasoning is carefully and rigorously performed, the result is a proof. Because of this intimate connection between understanding and proof, students should be exposed to both informal and formal proofs in many areas of mathematics. They should construct elementary proofs as a part of the development of their reasoning skills.

Mathematical Reasoning (continued)

Standard I: High school graduates will understand the meaning of logical implication, including the roles of hypotheses and conclusion in a deduction. In particular, they will understand that saying a second statement is a consequence of a first is different from saying that the first statement is a consequence of the second. High school graduates will test general assertions with examples. They will recognize why one counterexample is enough to show that a statement is false and they will understand that the truth of a general assertion in a few cases does not assure its truth in all cases.

In not only scientific but also political contexts, we all need to be able to understand the structure of arguments purporting to draw conclusions from hypotheses and data. The fallacy that an observation will necessarily be repeated in a similar situation, or that because a statement is true within a limited range of experience that it is true in all cases, can lead to erroneous conclusions.

Exemplars for Mathematical Reasoning Standard I:

A parking space is in front of a signpost with two signs. One sign says "2 hour Parking (M-Th), Except with Permit" and the other says "No Parking on Fridays."

Which of the following are then true?

- If you don't have a permit and it is Saturday, you can park for more than two hours.
- If you can park for more than two hours on a Wednesday, then you must have a permit.
- If you have a permit, then you can park for two hours on Fridays.
- If you don't have a permit, and you can park for more than two hours, then it must be a weekend.

Which of the following are true, and which are false? For which of these is the first condition a consequence of the second?

- If $x=5$, then $x^2-4x-5=0$.
- If $x^2-4x-5=0$, then $x=5$.
- If the world record for the 200 meter dash is broken at the Olympics, then the Olympic record time for the event is broken.
- If the diagonals of a quadrilateral bisect each other, then the quadrilateral is a square.

Provide an example to show the falsehood of the statement:

$$\sqrt{a^2 + b^2} < a + b \text{ whenever } a \geq 0 \text{ and } b \geq 0.$$

Consider the claim: Whenever n is a whole number, $2^{2^n} + 1$ is a prime number (a number whose only whole number divisors are 1 and itself).

A student verified that this is true when $n=0, 1, 2,$ and 3 . Has the student proved that the claim is true? Explain your answer.

Mathematical Reasoning (continued)

Standard II: High school graduates will experiment, identify mathematical patterns, and use the patterns to answer questions.

This process is key for solving new problems, both mathematical and non-mathematical. High school mathematical instruction cannot anticipate every mathematical problem graduates will see, and so they need techniques for solving the unanticipated problems.

Exemplars for Mathematical Reasoning Standard II:

Draw the next three figures in the sequence:



What should the 75th figure in the sequence look like?

A Pythagorean triple is a set of three whole numbers that are the lengths of the sides of a right triangle. The following are Pythagorean triples:

(3,4,5), (5,12,13), (7,24,25), (9,40,41), (11,60,61)

Use any patterns you can identify to find the next Pythagorean triple in this sequence.

Standard III: High school graduates will recognize errors in simple chains of reasoning. They will clearly present and explain simple chains of reasoning in abstract contexts involving several steps.

To answer most questions using reasoning, several steps are required. In order to examine that reasoning, we must analyze it in its component steps, and understand each of those steps.

Exemplars for Mathematical Reasoning Standard III:

In a dark room there is a drawer containing 24 red socks and 24 blue socks. What is the smallest number of socks you must take from the drawer to be sure that you have a pair of socks of the same color? Explain your answer fully. What is the smallest number of socks you must take from the drawer to be sure that you have a pair of socks of different colors? Explain your answer fully.

A rectangular sheet of paper has been folded in half, and then folded in half again. After this, the folded paper is rectangular in shape, 4" by 5". What could the dimensions of the full sheet of paper be? Explain.